The affine Hecke algebra

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1 The affine Hecke algebra

1.1 The alcove walk algebra

Fix notations for the Weyl group W, the extended affine Weyl group $\widetilde{W}$, and their action on $\Omega \times \mathfrak{h}_{\mathbb{R}}^*$ as in Section 2. Label the walls of the alcoves so that the fundamental alcove has walls labeled $0, 1, \ldots, n$ and the labeling is $\widetilde{W}$-equivariant (see the picture in (2.12)).

The *periodic orientation* is the orientation of the walls of the alcoves given by

$$\text{setting the positive side of } H_{\alpha,j} \qquad \text{to be} \qquad \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha^\vee \rangle > j\}. \tag{1.1}$$

This is an orientation of the walls of the alcoves such that if $\triangle$ is an alcove and $\lambda \in P$ then

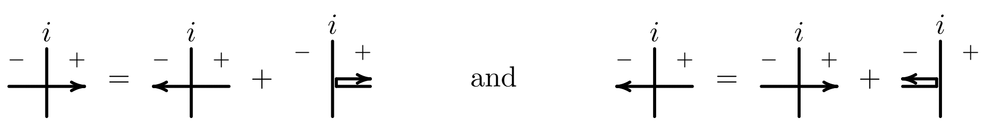
$$\text{the walls of } \lambda + \triangle \text{ have the same orientation as the walls of } \triangle.$$

Let $\mathbb{K}$ be a field. Use the notations for elements of Ω as in (2.10). The *alcove walk algebra* is the algebra over $\mathbb{K}$ given by generators $g \in \Omega$ and

$(1 \le i \le n)$

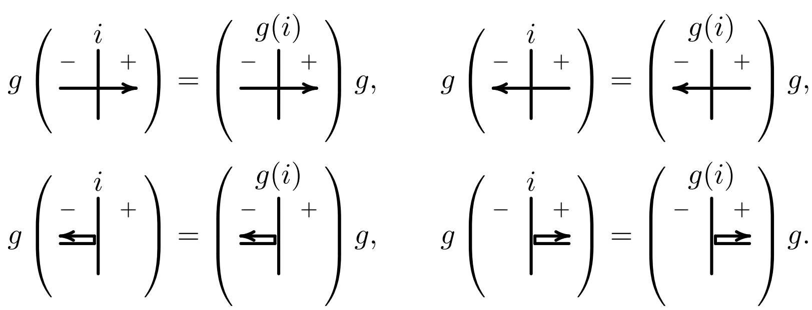
positive i-crossing negative i-crossing positive i-fold negative i-fold

with relations (straightening laws)



(1.2)

and



Viewing the product as concatenation each word in the generators can be represented as a sequence of arrows, with the first arrow having its head or its tail in the fundamental alcove. An *alcove walk* is a word in the generators such that,

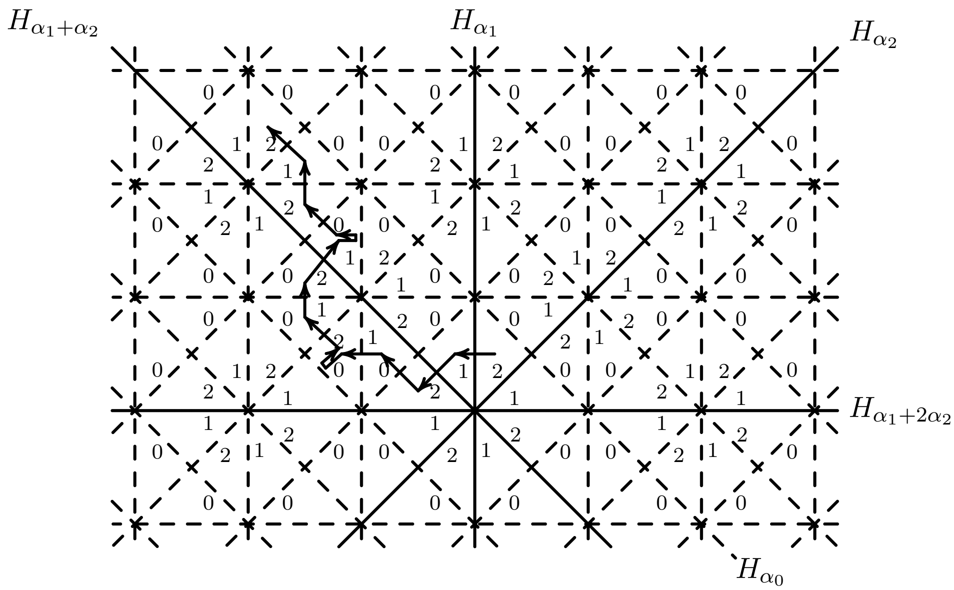
(a) the tail of the first step is in the fundamental alcove A,

(b) at every step, the head of each arrow is in the same alcove as the tail of the next arrow.

The *type* of a walk p is the sequence of labels on the arrows. Note that, if $w \in \widetilde{W}$ then

$$\ell(w) = \text{length of a minimal length walk from } A \text{ to } wA. \tag{1.3}$$

For example, in type C_2,



is an alcove walk p of type $(1,2,0,1,0,2,1,2,1,0,1,2)$ with two folds. Using the notation

$$\begin{array}{llll} c_i^+ & \text{for a positive } i\text{-crossing}, & f_i^+ & \text{for a positive } i\text{-fold}, \\ c_i^- & \text{for a negative } i\text{-crossing}, & f_i^- & \text{for a negative } i\text{-fold}, \end{array} \tag{1.4}$$

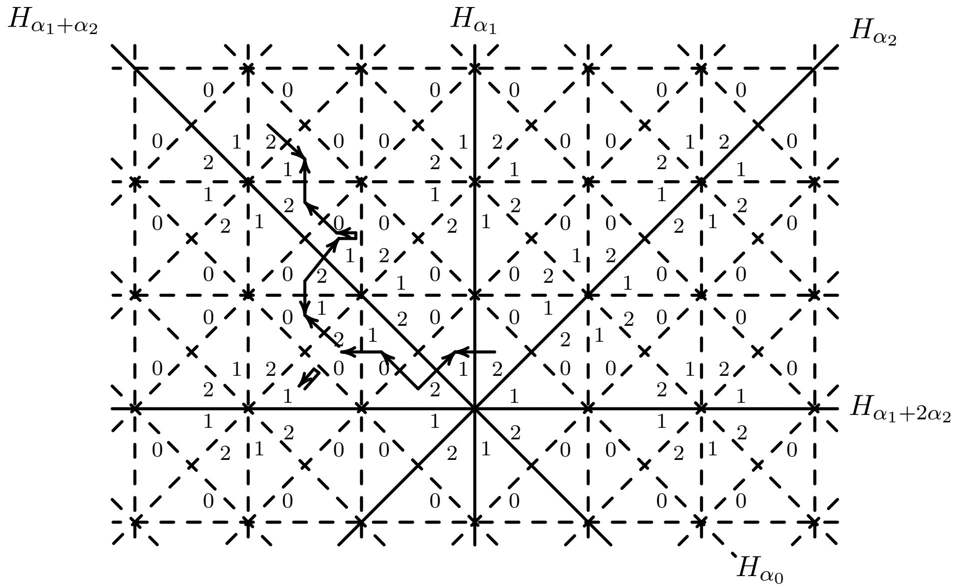
the walk in the picture is $c_1^- c_2^- c_0^+ c_1^- f_0^+ c_2^+ c_1^+ c_2^+ f_1^- c_0^+ c_1^+ c_2^+$.

The proof of the following lemma is straightforward following the scheme indicated by the example which follows.

Lemma 1.1. *The set of alcove walks is a basis of the alcove walk algebra.*

For example, in type C_2, a product of the generators which is not a walk is

$$c_1^- c_2^+ c_0^+ c_1^- f_0^- c_2^+ c_1^- c_2^+ f_1^- c_0^+ c_1^+ c_2^-,$$



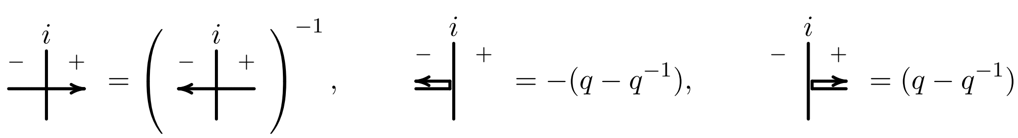
but, by first applying relations $f_i^{\mp} = -f_i^{\pm}$ and then working left to right applying the relations $c_i^{\pm} = c_i^{\mp} + f_i^{\pm}$, gives

$$\begin{aligned}
c_1^- c_2^+ c_0^+ c_1^- f_0^- c_2^+ c_1^- c_2^+ f_1^- c_0^+ c_1^+ c_2^- &= -(c_1^- c_2^+ c_0^+ c_1^- f_0^+ c_2^+ c_1^- c_2^+ f_1^- c_0^+ c_1^+ c_2^-) \\
&= -(c_1^-(c_2^- + f_2^+) c_0^+ c_1^- f_0^+ c_2^+ c_1^- c_2^+ f_1^- c_0^+ c_1^+ c_2^-) \\
&= -(c_1^-(c_2^- + f_2^+) c_0^+ c_1^- f_0^+ c_2^+ (c_1^+ + f_1^-) c_2^+ f_1^- c_0^+ c_1^+ c_2^-) \\
&= -\big(c_1^-(c_2^- + f_2^+) c_0^+ c_1^- f_0^+ c_2^+ (c_1^+ + f_1^-) c_2^+ f_1^- c_0^+ c_1^+ (c_2^+ + f_2^-)\big)
\end{aligned}$$

and every term in the expansion of this expression is an alcove walk.

1.2 The affine Hecke algebra

Fix an invertible element $q \in \mathbb{K}$. The *affine Hecke algebra* $\tilde{H}$ is the quotient of the alcove walk algebra by the relations



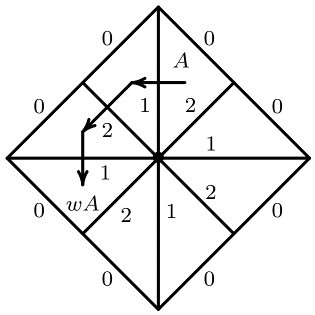
and

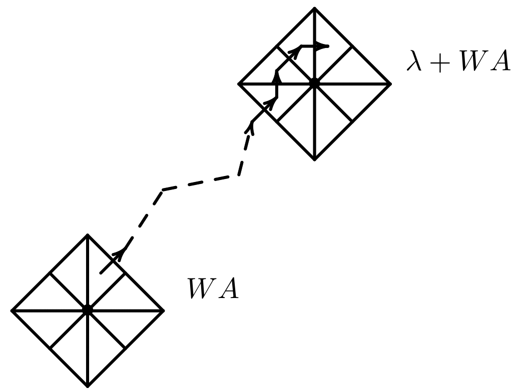
$$p = p' \qquad \text{if } p \text{ and } p' \text{ are nonfolded walks with } \mathrm{end}(p) = \mathrm{end}(p'), \tag{1.5}$$

where $\mathrm{end}(p)$ is the final alcove of p. Conceptually, the affine Hecke algebra only remembers the ending alcove of a walk (and some information about the folds) and forgets how it got to its destination.

For $w \in W$ and $\lambda \in P$ define elements

$$\begin{aligned}
T_{w^{-1}}^{-1} &= (\text{image in } \tilde{H} \text{ of a minimal length alcove walk from } A \text{ to } wA), \\
X^{\lambda} &= (\text{image in } \tilde{H} \text{ of a minimal length alcove walk from } A \text{ to } t_\lambda A).
\end{aligned}$$

$T_{w^{-1}}^{-1} =$ 

$X^\lambda =$ 

The following proposition shows that the alcove walk definition of the affine Hecke algebra coincides with the standard definition by generators and relations (see [IM] and [Lu]). A consequence of the proposition is that

$$\begin{array}{lll} \text{the } \textit{finite Hecke algebra}, & H = \mathrm{span}\{T_{w^{-1}}^{-1} \mid w \in W\}, & \text{and} \\ \text{the Laurent polynomial ring}, & \mathbb{K}[P] = \mathrm{span}\{X^\lambda \mid \lambda \in P\}, & \end{array} \tag{1.6}$$

are subalgebras of $\tilde{H}$.

Proposition 1.2. *Let $g \in \Omega$, $\lambda, \mu \in P$, $w \in W$ and $1 \le i \le n$. Let φ be the element of R^+ such that $H_{\alpha_0} = H_{\varphi,1}$ is the wall of A which is not a wall of C and let s_φ be the reflection in H_φ. Let w_0 be the longest element of W. The following identities hold in $\tilde{H}$.*

(a) $X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda$.

(b) $T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } \ell(s_i w) > \ell(w), \\ T_{s_i w} + (q - q^{-1}) T_w, & \text{if } \ell(s_i w) < \ell(w). \end{cases}$

(c) If $\langle \lambda, \alpha_i^\vee \rangle = 0$ *then* $T_{s_i} X^\lambda = X^\lambda T_{s_i}$.

(d) If $\langle \lambda, \alpha_i^\vee \rangle = 1$ *then* $T_{s_i} X^{s_i\lambda} T_{s_i} = X^\lambda$.

(e) $T_{s_i} X^\lambda = X^{s_i\lambda} T_{s_i} + (q - q^{-1}) \dfrac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}$.

(f) $T_{s_0} T_{s_\varphi} = X^\varphi$.

(g) $X^{\omega_i} = g T_{w_0 w_i}$*, where the action of g on A sends the origin to ω_i and w_i is the longest element of the stabilizer W_{ω_i} of ω_i in W.*

Proof. Use notations for alcove walks as in (3.4).

(a) If p_λ is a minimal length walk from A to $t_\lambda A$ and p_μ is a minimal length walk from from A to $t_\mu A$ then

$$p_\lambda p_\mu \text{ and } p_\mu p_\lambda \text{ are both nonfolded walks from } A \text{ to } t_{\lambda+\mu} A.$$

Thus the images of $p_\lambda p_\mu$ and $p_\mu p_\lambda$ are equal in $\tilde{H}$.

(b) If $\ell(ws_i) > \ell(w)$ and p_w is a minimal length walk from A to wA then

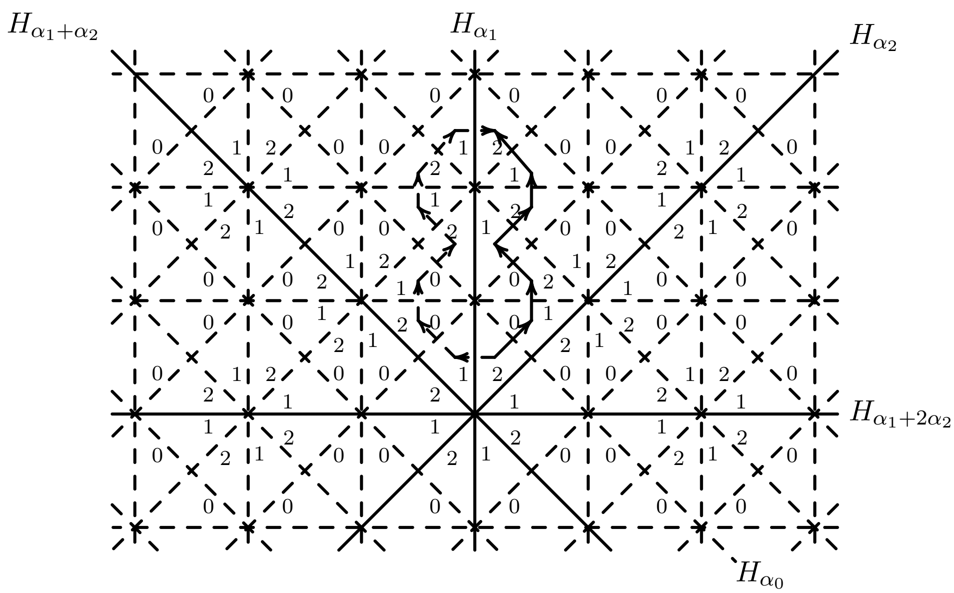
$$p_{ws_i} = p_w c_i^- \text{ is a minimal length walk from } A \text{ to } ws_i A.$$

and so $T_{s_i w^{-1}}^{-1} = T_{ws_i{}^{-1}}^{-1} = T_{w^{-1}}^{-1} T_{s_i}^{-1} = (T_{s_i} T_{w^{-1}})^{-1}$ in $\tilde{H}$. Taking inverses gives the first result, and the second follows by switching w and ws_i and using the relation $T_{s_i}^{-1} = T_{s_i} - (q - q^{-1})$ which follows from (3.2) and (3.5).

(c) Let p_λ be a minimal length alcove walk from A to $t_\lambda A$. If $\langle \lambda, \alpha_i^\vee \rangle = 0$ then H_{α_i} is a wall of $t_\lambda A$ and $s_i\lambda = \lambda$ and

$$c_i^- p_\lambda c_i^+ \text{ is a nonfolded walk from } A \text{ to } t_\lambda A.$$

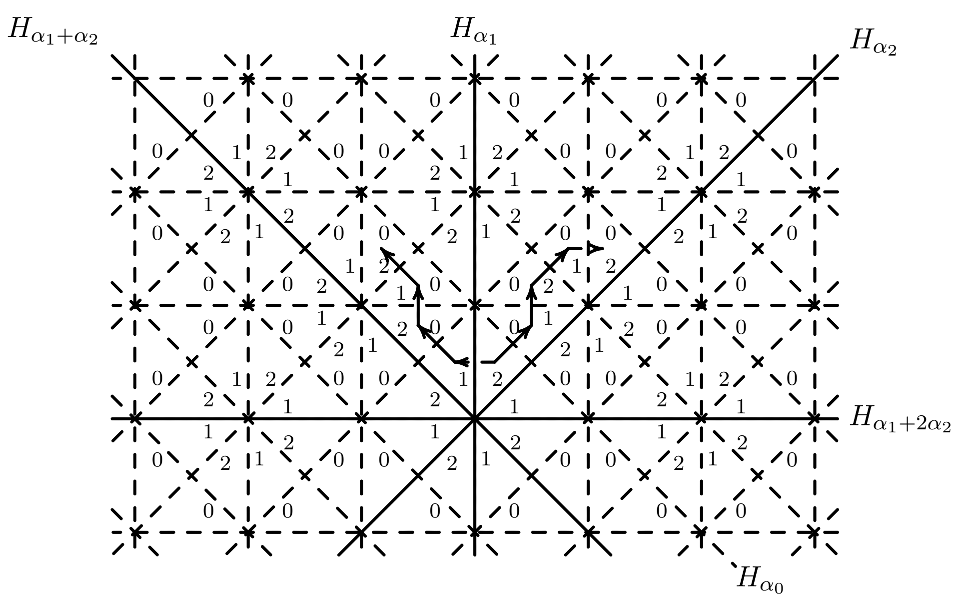
Thus $T_{s_i}^{-1} X^\lambda T_{s_i} = X^\lambda = X^{s_i\lambda}$ in $\tilde{H}$.



(d) Let p_λ be a minimal length walk from A to $t_\lambda A$. If $\langle \lambda, \alpha_i^\vee \rangle = 1$ then there is a minimal length walk from A to $t_\lambda A$ of the form $p_\lambda = p_{t_\lambda s_i} c_i^+$ where $p_{t_\lambda s_i}$ is minimal length walk from A to $t_\lambda s_i A$. Then

$$c_i^- p_{t_\lambda s_i} \text{ is a minimal length walk from } A \text{ to } t_{s_i\lambda} A.$$

Thus $T_{s_i}^{-1}(X^\lambda T_{s_i^{-1}}) = X^{s_i\lambda}$ in $\tilde{H}$.



(e) Note that (c) and (d) are special cases of (e). If the statement of (e) holds for λ then, by multiplying on the left by $X^{-s_i\lambda}$ and on the right by $X^-\lambda$, it holds for $-\lambda$, If the statement (e) holds for λ and μ then it holds for $\lambda+\mu$ since

$$\begin{aligned} T_{s_i}X^\lambda X^\mu &= \left(X^{s_i\lambda}T_{s_i} + (q-q^{-1})\frac{X^\lambda - X^{s_i\lambda}}{1-X^{-\alpha_i}}\right)X^\mu \\ &= X^{s_i\lambda}\left(X^{s_i\mu}T_{s_i} + (q-q^{-1})\frac{X^\mu - X^{s_i\mu}}{1-X^{-\alpha_i}}\right) + (q-q^{-1})\left(\frac{X^\lambda - X^{s_i\lambda}}{1-X^{-\alpha_i}}\right)X^\mu \\ &= X^{s_i(\lambda+\mu)}T_{s_i} + (q-q^{-1})\frac{X^{\lambda+\mu} - X^{s_i(\lambda+\mu)}}{1-X^{-\alpha_i}}. \end{aligned}$$

Thus, to prove (e) it is sufficient to verify (c) and (d), which has already been done.
(f) Let p_{s_φ} be a minimal length walk from $s_\varphi A$ to A, then

$$p_\varphi = c_0^+ p_{s_\varphi} \text{ is a minimal length walk from } A \text{ to } t_\varphi A.$$

Thus $T_0T_{s_\varphi} = X^\varphi$ in $\tilde{H}$.
(g) If $p_{w_0w_i}$ is a minimal length walk from w_iw_0A to A then

$$p_{\omega_i} = gp_{w_0w_i} \text{ is a minimal length walk from } A \text{ to } t_{\omega_i}A.$$

Thus $X^{\omega_i} = gT_{w_0w_i}$ in $\tilde{H}$. For example, in type C_2, $w_0 = s_2s_1s_2s_1$ and there is one element g in Ω such that $g \neq 1$ for which $g\omega_2 = 0$ and $w_2 = s_1$ so that $w_0w_2 = s_2s_1s_2$. □

The sets

$$\{T_{w^{-1}}^{-1}X^\lambda \mid w \in W, \lambda \in P\} \qquad \text{and} \qquad \{X^\mu T_{v^{-1}}^{-1} \mid \mu \in P, v \in W\} \tag{1.7}$$

are bases of $\tilde{H}$. If p is an alcove walk then the *weight* of p and the *final direction* of p are

$$\mathrm{wt}(p) \in P \text{ and } \varphi(p) \in W \qquad \text{such that} \qquad p \text{ ends in the alcove } \mathrm{wt}(p) + \varphi(p)A. \tag{1.8}$$

Let

$$\begin{aligned} f^-(p) &= (\text{number of negative folds of } p), \\ f^+(p) &= (\text{number of positive folds of } p), \qquad \text{and} \\ f(p) &= (\text{total number of folds of } p). \end{aligned} \tag{1.9}$$

The following theorem provides a combinatorial formulation of the transition matrix between the bases in (3.7). It is a q-version of the main result of [LP] and an extension of Corollary 6.1 of [Sc].

Theorem 1.3. *Use notations as in (3.4). Let $\lambda \in P$ and $w \in W$. Fix a minimal length walk $p_w = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^-$ from A to wA and a minimal length walk $p_\lambda = c_{j_1}^{\epsilon_1}\cdots c_{j_s}^{\epsilon_s}$ from A to $t_\lambda A$. Then, with notations as in (3.8) and (3.9),*

$$T_{w^{-1}}^{-1}X^\lambda = \sum_p (-1)^{f^-(p)}(q-q^{-1})^{f(p)}X^{\mathrm{wt}(p)}T_{\varphi(p)^{-1}}^{-1},$$

where the sum is over all alcove walks $p = c_{i_1}^-\cdots c_{i_r}^- p_{j_1}\cdots p_{j_s}$ such that p_{j_k} is either $c_{j_k}^{\epsilon_k}$, $c_{j_k}^{-\epsilon_k}$ or $f_{j_k}^{\epsilon_k}$.

Proof. The product $p_w p_\lambda = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^- c_{j_1}^{\epsilon_1} \cdots c_{j_s}^{\epsilon_s}$ may not necessarily be walk, but its straightening produces a sum of walks, and this decomposition gives the formula in the statement. □

Remark 1.4. The *initial direction* $\iota(p)$ and the *final direction* $\varphi(p)$ of an alcove walk p appear naturally in Theorem 3.3. These statistics also appear in the Pieri-Chevalley formula in the K-theory of the flag variety (see [PR], [GR], [Br] and [LP]).

Remark 1.5. In Theorem 3.3, for certain λ the walk p_λ may be chosen so that all the terms in the expansion of $T_{w^{-1}}^{-1} X^\lambda$ have the same sign. For example, if λ is dominant, then p_λ can be taken with all $\epsilon_k = +$, in which case all folds which appear in the straightening of $p_w p_\lambda$ will be positive folds and so all terms in the expansion will be positive. If λ is antidominant then p_λ can be taken with all $\epsilon_k = -$ and all terms in the expansion will be negative. This fact gives positivity results for products in the cohomology and the K-theory of the flag variety (see [PR], [Br]).

Remark 1.6. The affine Hecke algebra $\tilde{H}$ has basis $\{X^\lambda T_{w^{-1}}^{-1} \mid \lambda \in P, w \in W\}$ in bijection with the alcoves in $\Omega \times \mathfrak{h}_{\mathbb{R}}^*$, where $X^\lambda T_{w^{-1}}^{-1}$ is the image in $\tilde{H}$ of a minimal length alcove walk from A to the alcove $\lambda + wA$. Changing the orientation of the walls of the alcoves changes the resulting basis in the affine Hecke algebra $\tilde{H}$. The orientation in (3.1) is the one such that

$$\text{the most negative point is } -\infty\rho\text{, deep in the chamber } w_0 C. \tag{1.10}$$

Another standard orientation is where

$$\text{the most negative point is the center of the fundamental alcove } A. \tag{1.11}$$

Using the orientation of the walls given by (3.11) produces the basis commonly denoted $\{T_w \mid w \in \widetilde{W}\}$ by taking T_w to be the image in $\tilde{H}$ of a minimal length alcove walk from A to $w^{-1}A$. Since $T_i^{-1} = T_i - (q - q^{-1})$ the transition matrix between the basis $\{X^\lambda T_{w^{-1}}^{-1} \mid \lambda \in P, w \in W\}$ and the basis $\{T_w \mid w \in \widetilde{W}\}$ is triangular.