Affine braid groups of classical type

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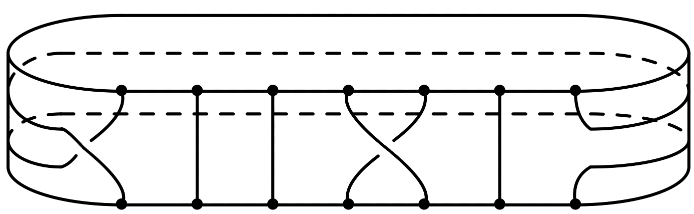
1 The type GL_n affine braid group

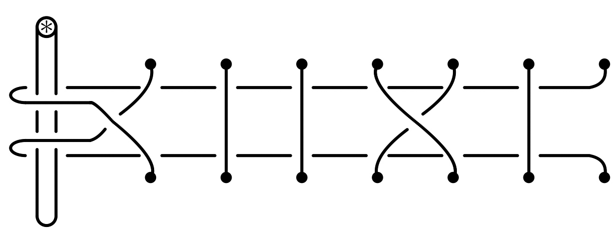
There are three equivalent ways of depicting affine braids

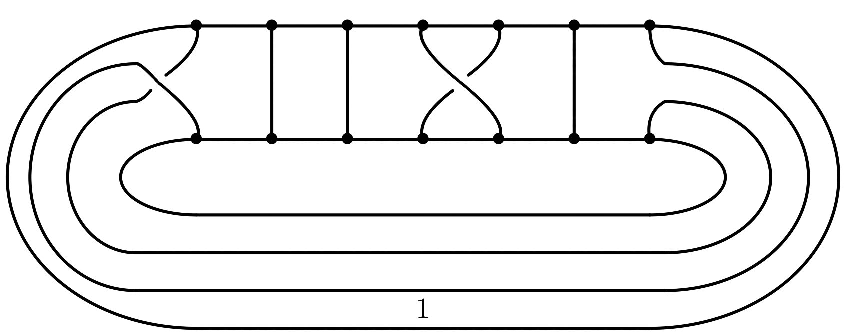
(a) As braids in a (slightly thickened) cylinder,

(b) As braids in a (slightly thickened) annulus,

(c) As braids with a flagpole.

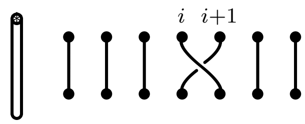


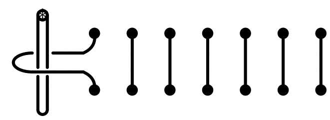




The multiplication is by placing one cylinder on top of another, placing one annulus inside another, or placing one flagpole braid on top of another. These are equivalent formulations since an annulus can be made into a cylinder by turning up the edges and a cylindrical braid can be made into a flagpole braid by putting a flagpole down the middle of the cylinder and pushing the pole over to the left so that the strings begin and end to its right.

The *affine braid group* is the group $\tilde{\mathcal{B}}_k$ formed by the affine braids with k strands. The affine braid group $\tilde{\mathcal{B}}_k$ can be presented by generators $T_1, T_2, \ldots, T_{k-1}$ and X^{ε_1}

$T_i =$ 

and $X^{\varepsilon_1} =$  (1.1)

with relations

(a) $T_iT_j = T_jT_i$, if $|i-j| > 1$,

(b) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $1 \le i \le k-2$,

(c) $X^{\varepsilon_1}T_1X^{\varepsilon_1}T_1 = T_1X^{\varepsilon_1}T_1X^{\varepsilon_1}$,

(d) $X^{\varepsilon_1}T_i = T_iX^{\varepsilon_1}$, for $2 \le i \le k-1$.

For $1 \le i \le k$ define

$$X^{\varepsilon_i} = T_{i-1}T_{i-2}\cdots T_2T_1X^{\varepsilon_1}T_1T_2\cdots T_{i-1} = PICTURE. \tag{1.2}$$

By drawing pictures of the corresponding affine braids it is easy to check that the X^{ε_i} all commute with each other and so $X = \langle X^{\varepsilon_i} \mid 1 \le i \le k\rangle$ is an abelian subgroup of $\tilde{\mathcal{B}}_k$. Let $L \cong \mathbb{Z}^k$ be the free abelian group generated by $\varepsilon_1, \ldots, \varepsilon_k$. Then

$$L = \{\lambda_1\varepsilon_1 + \cdots + \lambda_k\varepsilon_k \mid \lambda_i \in \mathbb{Z}\} \qquad \text{and} \qquad X = \{X^\lambda \mid \lambda \in L\}, \tag{1.3}$$

where $X^\lambda = (X^{\varepsilon_1})^{\lambda_1}(X^{\varepsilon_2})^{\lambda_2}\cdots(X^{\varepsilon_k})^{\lambda_k}$, for $\lambda \in L$.

For $1 \le i \le k$ define

$$X^{\varepsilon_i} = T_{i-1}T_{i-2}\cdots T_2T_1X^{\varepsilon_1}T_1T_2\cdots T_{i-1} = PICTURE, \tag{1.4}$$

By drawing pictures of the corresponding affine braids it is easy to check that the X^{ε_i} all commute with each other and so $X = \langle X^{\varepsilon_i} \mid 1 \le i \le k\rangle$ is an abelian subgroup of $\tilde{\mathcal{B}}_k$. The free abelian group generated by $\varepsilon_1, \ldots, \varepsilon_k$ is $\mathbb{Z}^k$ and

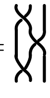
$$X = \{X^\lambda \mid \lambda \in \mathbb{Z}^k\} \quad \text{where} \quad X^\lambda = (X^{\varepsilon_1})^{\lambda_1}(X^{\varepsilon_2})^{\lambda_2}\cdots(X^{\varepsilon_k})^{\lambda_k}, \tag{1.5}$$

for $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_k\varepsilon_k$ in $\mathbb{Z}^k$.

Let U be a quasitriangular Hopf algebra Let M and V be U-modules such that the operators $\check{R}_{MV}$, $\check{R}_{VM}$ and $\check{R}_{VV}$ are well defined. Define $\check{R}_i$, $1 \le i \le k-1$, and $\check{R}_0^2$ in $\mathrm{End}_U(M \otimes V^{\otimes k})$ by

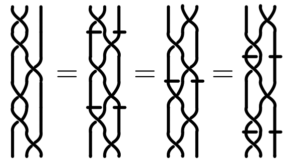
$$\check{R}_i = \mathrm{id}_M \otimes \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}_V^{\otimes(k-i-1)} \qquad \text{and} \qquad \check{R}_0^2 = (\check{R}_{MV}\check{R}_{VM}) \otimes \mathrm{id}_V^{\otimes(k-1)}.$$

Then the braid relations

$\check{R}_i\check{R}_{i+1}\check{R}_i =$  $=$  $= \check{R}_{i+1}\check{R}_i\check{R}_{i+1}$

and

$$\check{R}_0^2\check{R}_1\check{R}_0^2\check{R}_1 =$$



$$= \check{R}_1\check{R}_0^2\check{R}_1\check{R}_0^2.$$

imply that there is a well defined map

$$\begin{array}{rccc} \Phi\colon & \tilde{\mathcal{B}}_k & \longrightarrow & \mathrm{End}_U(M\otimes V^{\otimes k}) & \\ & T_i & \longmapsto & \check{R}_i, & 1\le i\le k-1, \\ & X^{\varepsilon_1} & \longmapsto & \check{R}_0^2, & \end{array}$$

which makes $M\otimes V^{\otimes k}$ into a right $\tilde{\mathcal{B}}_k$ module. Note that

$$\Phi(X^{\varepsilon_i}) = \check{R}_{M\otimes V^{\otimes(i-1)},V}\check{R}_{V,M\otimes V^{\otimes i}}$$

and thus, by (???), the eigenvalues of $\Phi(X^{\varepsilon_i})$ are related to the eigenvalues of the Casimir.

1.1 The $\tilde{\mathcal{B}}_k$ module $M\otimes V^{\otimes k}$

Let U be a quasitriangular Hopf algebra. Let M and V be a U-modules such that $\check{R}_{MV}$ and $\check{R}_{VV}$ are well defined operators. Define $\check{R}_i$, $1\le i\le k-1$, and $\check{R}_0^2$ in $\mathrm{End}_{U_h\mathfrak{g}}(M\otimes V^{\otimes k})$ by

$$\check{R}_i = \mathrm{id}_M\otimes\mathrm{id}_V^{\otimes(i-1)}\otimes\check{R}_{VV}\otimes\mathrm{id}_V^{\otimes(k-i-1)} \qquad\text{and}\qquad \check{R}_0^2 = (\check{R}_{MV}\check{R}_{VM})\otimes\mathrm{id}_V^{\otimes(k-1)}.$$

Proposition 1.1. *The map defined by*

$$\begin{array}{rccc} \Phi\colon & \tilde{\mathcal{B}}_k & \longrightarrow & \mathrm{End}_{U_h\mathfrak{g}}(M\otimes V^{\otimes k}) & \\ & T_i & \longmapsto & \check{R}_i, & 1\le i\le k-1, \\ & X^{\varepsilon_1} & \longmapsto & \check{R}_0^2, & \end{array}$$

makes $M\otimes V^{\otimes k}$ into a right $\tilde{\mathcal{B}}_k$ module.

Proof. It is necessary to show that

(a) $\check{R}_i\check{R}_j = \check{R}_j\check{R}_i$, if $|i-j|>1$,

(b) $\check{R}_0^2\check{R}_i = \check{R}_i\check{R}_0^2$, $i>2$,

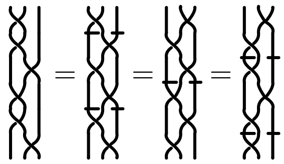
(c) $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}$, $1\le i\le k-2$,

(d) $\check{R}_0^2\check{R}_1\check{R}_0^2\check{R}_1 = \check{R}_1\check{R}_0^2\check{R}_1\check{R}_0^2$.

The relations (a) and (b) follow immediately from the definitions of $\check{R}_i$ and $\check{R}_0^2$ and (c) is a particular case of the braid relation (???). The relation (d) is also a consequence of the braid relation:

$$\begin{aligned} \check{R}_0^2\check{R}_1\check{R}_0^2\check{R}_1 &= (\check{R}_{MV}\check{R}_{VM}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{VV})(\check{R}_{MV}\check{R}_{VM}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{VV}) \\ &= (\check{R}_{MV}\otimes\mathrm{id})\underbrace{(\mathrm{id}\otimes\check{R}_{MV})(\check{R}_{VV}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{VM})}(\check{R}_{VM}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{VV}) \\ &= \underbrace{(\mathrm{id}\otimes\check{R}_{VV})(\check{R}_{MV}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{MV})}\,\underbrace{(\check{R}_{VV}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{VM})(\check{R}_{VM}\otimes\mathrm{id})} \\ &= (\mathrm{id}\otimes\check{R}_{VV})(\check{R}_{MV}\,\overbrace{\check{R}_{VM}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}_{VV})(\check{R}_{MV}}\,\check{R}_{VM}\otimes\mathrm{id}) \\ &= \check{R}_1\check{R}_0^2\check{R}_1\check{R}_0^2, \end{aligned}$$

or equivalently,

$$\check{R}_0^2\check{R}_1\check{R}_0^2\check{R}_1 =$$

$$= \check{R}_1\check{R}_0^2\check{R}_1\check{R}_0^2.$$

□

1.2 Schur functors

Fix a U module V and a weight λ in $\mathfrak{h}^*$ and let $M(\lambda)$ be the Verma module of highest weight λ. The *Schur functor* from U-modules to $\tilde{B}_k$-modules is the functor $F_{\lambda,V}$ given by

$$F_{\lambda,V}(M) = \mathrm{Hom}_U(M(\lambda), M \otimes V^{\otimes k}). \tag{1.6}$$

The functors $F_{\lambda,V}$ are interesting whenever they are well defined. Of particular importance are the $\tilde{\mathcal{B}}_k$ modules

$$\mathcal{M}^{\lambda/\mu} = F_{\lambda,V}(M(\mu)) \qquad \text{and} \qquad \mathcal{L}^{\lambda/\mu} = F_{\lambda,V}(L(\mu)). \tag{1.7}$$

Since the image of $M(\lambda)$ under a U-module homomorphism is determined by the image of a generating highest weight vector, the $\tilde{\mathcal{B}}_k$ module $F_{\lambda,V}(M)$ can be identified with the vector space of highest weight vectors of weight λ in $M \otimes V^{\otimes k}$. The functor $F_{\lambda,V}$ is the composition of two functors: the functor $\cdot \otimes V^{\otimes k}$ and the functor $\mathrm{Hom}_U(M(\lambda), \cdot)$. The first is exact when V is finite dimensional and the second is exact when λ is integrally dominant, because these are cases when V is flat and $M(\lambda)$ is projective, see [Jz, p. 72]. More generally one should analyze all the functors

$$F^i_{\lambda,V}(M) = \mathrm{Ext}^i_U(M(\lambda), M \otimes V^{\otimes k}).$$

1.3 Restriction from $\tilde{\mathcal{B}}_k$ to $\mathcal{B}_k$

Proposition 1.2. *The braid group $\mathcal{B}_k$ is the quotient of the affine braid group by the relation $X^{\varepsilon_1} = 1$ and so the modules $\mathcal{L}^{\nu/0}$ are $\mathcal{B}_k$-modules. Let P^+ be the set of dominant integral weights. Define the tensor product multiplicities $c^\lambda_{\mu\nu}$, $\lambda, \mu, \nu \in P^+$, by the $U_h\mathfrak{g}$-module decompositions*

$$L(\mu) \otimes L(\nu) \cong \bigoplus_{\lambda \in P^+} L(\lambda)^{\oplus c^\lambda_{\mu\nu}}.$$

Then

$$\mathrm{Res}^{\tilde{\mathcal{B}}_k}_{\mathcal{B}_k}(\mathcal{L}^{\lambda/\mu}) = \bigoplus_{\nu \in P^+} (\mathcal{L}^{\nu/0})^{\oplus c^\lambda_{\mu\nu}}.$$

Proof. Let us abuse notation slightly and write sums instead of direct sums. Then, as a $(U_h\mathfrak{g}, \mathcal{B}_k)$ bimodule

$$L(\mu) \otimes V^{\otimes k} = \sum_\lambda L(\lambda) \otimes \mathcal{L}^{\lambda/\mu},$$

where $\mathcal{L}^{\lambda/\mu} = F_\lambda(L(\mu))$. As a $(U_h\mathfrak{g}, \mathcal{B}_k)$ bimodule

$$L(\mu) \otimes V^{\otimes k} = L(\mu) \otimes \left(\sum_\nu L(\nu) \otimes \mathcal{L}^{\nu/0} \right) = \sum_{\lambda,\nu} c^\lambda_{\mu\nu} L(\lambda) \otimes \mathcal{L}^{\nu/0}.$$

Comparing coefficients of $L(\lambda)$ in these two identities yields the formula in the statement. □

1.4 Quantum traces

For $z \in End(M)$ such that$z$ commutes with $e^{h\rho}$ define the *quantum trace* of z by

$$\mathrm{qtr}(z) = \mathrm{tr}(e^{h\rho} z).$$

The *quantum dimension* of M is

$$\mathrm{qdim}(M) = \mathrm{qtr}(\mathrm{id}_M).$$

If M is a semisimple U-module and $z \in \mathrm{End}_U(M)$ then

$$\mathrm{tr}_q(z) = \sum_{\lambda \in \hat{M}} \dim_q(L(\lambda))\chi_M^\lambda(z), \qquad \text{since} \qquad M \cong \bigoplus_{\lambda \in \hat{M}} L(\lambda) \otimes \mathcal{Z}^\lambda,$$

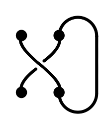
as a $(U, \mathcal{Z})$-bimodule, where $\mathcal{Z} = \mathrm{End}_U(M)$, $L(\lambda)$ are simple U-modules and $\mathcal{Z}^\lambda$ are the simple $\mathcal{Z}$ modules. There are natural injections

$$\begin{array}{ccc} \mathrm{End}_{U_0}(M) & \hookrightarrow & \mathrm{End}_{U_0}(M \otimes V) \\ z & \longmapsto & z \otimes \mathrm{id}_V \end{array}$$

Proposition 1.3. *Then*

$$\mathrm{qtr}_{M\otimes V}(z) = \mathrm{qdim}(V)\mathrm{qtr}_M(z) \qquad \textit{and} \qquad \mathrm{qtr}_{M\otimes V}(z\check{R}_{MV}) = \alpha \mathrm{qtr}_M(z),$$

where $\alpha =$????.



By Proposition (3.7) (a) it is enough to show that $\check{e}_2\check{R}\check{e}_2 = (\dim_q(V))^{-1}v(\lambda)^{-1}\check{e}_2$ as elements of $\mathrm{End}_U(V \otimes V \otimes V^*)$. Let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be a dual basis in V^*. It follows from the identities (2.5), (2.6) and (2.7) that if $\mathcal{R} = \sum_i a_i \otimes b_i$ and $(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = \sum_j c_j \otimes d_j$, then

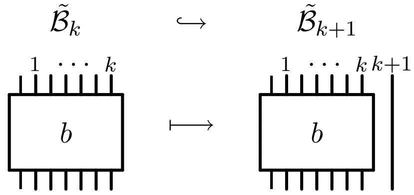
$$\sum_i b_i S^2(a_i) = \sum_j d_j S(c_j) = \sum_j S^{-1}(d_j)c_j = u^{-1}.$$

Let $x, y \in V$ and let $\phi \in V^*$. Then,

$$\begin{aligned}
\check{e}_2 \check{R} \check{e}_2 (x \otimes y \otimes \phi) &= (\dim_q(V))^{-1} \langle \phi, v^{-1}uy \rangle \check{e}_2 \check{R} \sum_k x \otimes e_k \otimes e^k \\
&= (\dim_q(V))^{-1} \langle \phi, v^{-1}uy \rangle \check{e}_2 \sum_{k,i} b_i e_k \otimes a_i x \otimes e^k \\
&= (\dim_q(V))^{-2} \langle \phi, v^{-1}uy \rangle \sum_{k,i,l} \langle e^k, v^{-1}ua_i x \rangle b_i e_k \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-2} \langle \phi, v^{-1}uy \rangle \sum_{i,l} (b_i v^{-1} u a_i x) \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-2} \langle \phi, v^{-1}uy \rangle \sum_{i,l} b_i S^2(a_i) v^{-1} u x \otimes e_l \otimes e^l \\
&= (\dim_q(V))^{-2} \langle \phi, v^{-1}uy \rangle \sum_{l} u^{-1} v^{-1} u x \otimes e_l \otimes e^l \\
&= \check{e}_2 (v^{-1} x \otimes y \otimes \phi) \\
&= (\dim_q(V))^{-1} v(\lambda)^{-1} \check{e}_2 (x \otimes y \otimes \phi).
\end{aligned}$$

1.5 Markov traces

A Markov trace on the affine braid group is a trace functional which respects the inclusions $\tilde{\mathcal{B}}_1 \subseteq \tilde{\mathcal{B}}_2 \subseteq \cdots$ where



(1.8)

More precisely, a *Markov trace* on the affine braid group with parameters $z, Q_1, Q_2, \ldots \in \mathbb{C}$ is a sequence of functions

$$\mathrm{mt}_k \colon \tilde{\mathcal{B}}_k \longrightarrow \mathbb{C} \qquad \text{such that}$$

(1) $\mathrm{mt}_1(1) = 1$,

(2) $\mathrm{mt}_{k+1}(b) = \mathrm{mt}_k(b)$, for $b \in \tilde{\mathcal{B}}_k$,

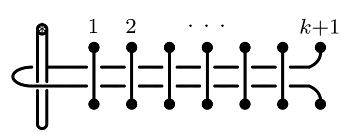
(3) $\mathrm{mt}_k(b_1 b_2) = \mathrm{mt}_k(b_2 b_1)$, for $b_1, b_2 \in \tilde{\mathcal{B}}_k$,

(4) $\mathrm{mt}_{k+1}(bT_k) = z\mathrm{mt}_k(b)$, for $b \in \tilde{\mathcal{B}}_k$,

(5) $\mathrm{mt}_{k+1}(b(\tilde{X}^{\varepsilon_{k+1}})^r) = Q_r \mathrm{mt}_k(b)$, for $b \in \tilde{\mathcal{B}}_k$,

where

$$\tilde{X}^{\varepsilon_{k+1}} = T_k T_{k-1} \cdots T_2 X^{\varepsilon_1} T_2^{-1} \cdots T_{k-1}^{-1} T_k^{-1} =$$



If M is a finite dimensional $U = U_h\mathfrak{g}$ module and $a \in \mathrm{End}_U(M)$ the *quantum trace* of a on M is the trace of the action of $e^{h\rho}a$ on M,

$$\mathrm{tr}_q(a) = \mathrm{Tr}(e^{h\rho}a, M), \qquad \text{and} \qquad \dim_q(M) = \mathrm{tr}_q(\mathrm{id}_M) = \mathrm{Tr}(e^{h\rho}, M) \tag{1.9}$$

is the *quantum dimension* of M.

Theorem 1.4. *Let $\mu, \nu \in P^+$ be dominant integral weights. Let $M = L(\mu)$ and $V = L(\nu)$ and let Φ_k be the representation of $\tilde{\mathcal{B}}_k$ defined in Proposition 3.5. Then the functions*

$$\begin{array}{rccc} \mathrm{mt}_k\colon & \tilde{\mathcal{B}}_k & \longrightarrow & \mathbb{C} \\ & b & \longmapsto & \dfrac{\mathrm{tr}_q(\Phi_k(b))}{\dim_q(M)\dim_q(V)^k} \end{array}$$

form a Markov trace on the affine braid group with parameters

$$z = \frac{q^{\langle \nu, \nu+2\rho\rangle}}{\dim_q(V)} \quad \textit{and} \quad Q_r = \sum_\lambda q^{r(\langle \lambda, \lambda+2\rho\rangle - \langle \mu, \mu+2\rho\rangle - \langle \nu, \nu+2\rho\rangle)} \frac{\dim_q(L(\lambda)) c_{\mu\nu}^\lambda}{\dim_q(L(\mu))\dim_q(L(\nu))},$$

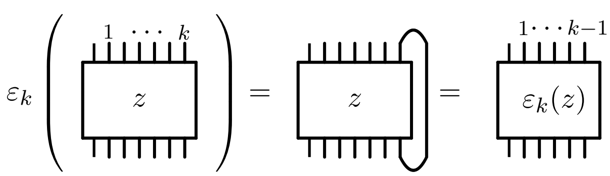
where the positive integers $c_{\mu\nu}^\lambda$ and the sum in the expression for Q_r are as in the tensor product decomposition

$$L(\mu) \otimes L(\nu) = \bigoplus_\lambda L(\lambda)^{\oplus c_{\mu\nu}^\lambda}.$$

Let $\varepsilon_k\colon \mathrm{End}_U(M \otimes V^{\otimes k}) \to \mathrm{End}_U(M \otimes V^{\otimes (k-1)})$ be given by

$$\varepsilon_k(z) = (\mathrm{id}_{M\otimes V^{\otimes(k-1)}} \otimes \check{e}) \circ (z \otimes \mathrm{id}) \qquad \text{where} \qquad \begin{array}{rccc} \check{e}\colon & V \otimes V^* & \longrightarrow & \mathbb{C} \\ & x \otimes \phi & \longmapsto & \dim_q(V)^{-1}\phi(e^{h\rho}x). \end{array} \tag{1.10}$$

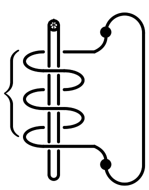
If V is simple then $\check{e}$ is the unique U-invariant projection onto the invariants in $V \otimes V^*$. Pictorially,



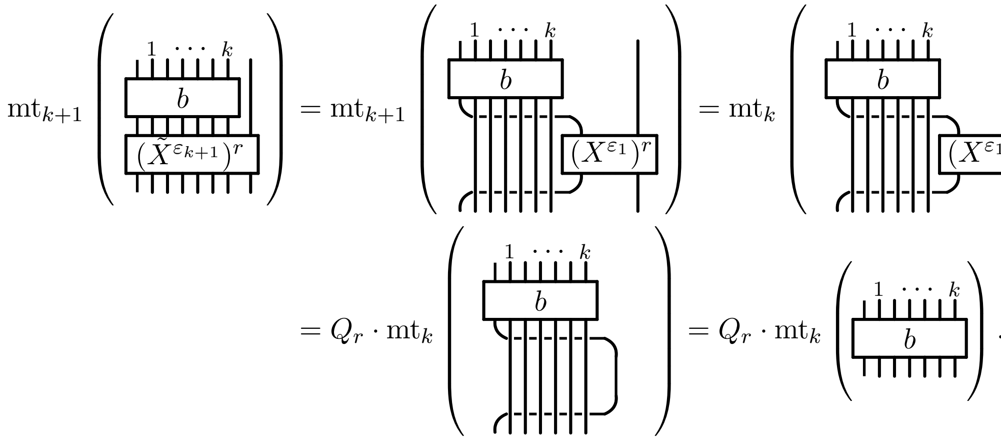
The argument of [LR] Theorem 3.10b shows that

$$\mathrm{mt}_k(b) = \mathrm{mt}_{k-1}(\varepsilon_{k-1}(b)), \qquad \text{if } b \in \tilde{\mathcal{B}}_k. \tag{1.11}$$

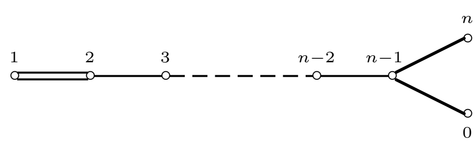
Since $\varepsilon_1((X^{\varepsilon_1})^r)$ is a $U_h\mathfrak{g}$-module homomorphism from M to M and, since M is simple, Schur's lemma implies that

r loops  $= \varepsilon_1((X^{\varepsilon_1})^r) = Q_r \cdot \mathrm{id}_M, \qquad$ for some $Q_r \in \mathbb{C}$.

Let $\tilde{R}_i = \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VM} \otimes \mathrm{id}_V^{(k+1)-i}$ so that $(\tilde{X}^{\varepsilon_{k+1}})^r = (\tilde{R}_k \cdots \tilde{R}_1)^{-1}(X^{\varepsilon_1})^r(\tilde{R}_k \cdots \tilde{R}_1)$. Then



2 Affine Hecke algebras of types B and C



$$\begin{aligned} &\alpha_1 = \varepsilon_1, && \alpha_1^\vee = 2\varepsilon_1, && \omega_1^\vee = \varepsilon_1 + \cdots + \varepsilon_n, \\ &\alpha_i = \varepsilon_i - \varepsilon_{i-1}, && \alpha_i^\vee = \varepsilon_i - \varepsilon_{i-1}, && \omega_i^\vee = \varepsilon_i + \cdots + \varepsilon_n, \end{aligned}$$

$$P^\vee = \sum_{i=1}^n \mathbb{Z}\varepsilon_i, \qquad \text{and} \qquad Q^\vee = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda + 1 \cdots + \lambda_n = 0 \bmod 2\}.$$

Then

$$\varphi = \varphi^\vee = \varepsilon_n + \varepsilon_{n-1} \qquad \text{and} \qquad s_\varphi = t_{n-1}t_n s_{n-1,n},$$

so that

$$s_\varphi = s_{n-1}s_{n-2}\cdots s_2s_1s_2\cdots s_{n-1}s_ns_{n-1}\cdots s_2s_1s_2\cdots s_{n-1}.$$

So

$$X^{\varepsilon_{n-1}+\varepsilon_n} = T_0T_{n-1}\cdots T_2T_1T_2\cdots T_{n-1}T_nT_{n-1}\cdots T_2T_1T_2\cdots T_{n-1}.$$

Next

$$\omega_n^\vee = \varepsilon_n, \qquad w_n = t_1\cdots t_{n-1}, \qquad \text{and} \qquad w_0 = t_1\cdots t_n,$$

so that

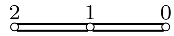
$$w_0w_n = t_n = s_ns_{n-1}\cdots s_2s_1s_2\cdots s_n$$

and

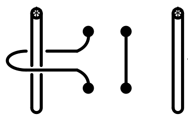
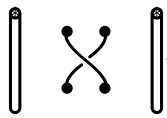
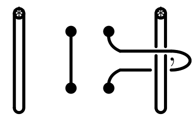
$$X^{\varepsilon_n} = \sigma T_nT_{n-1}\cdots T_2T_1T_2\cdots T_n.$$

Then $X^{\varepsilon_i} = T_{i+1}^{-1}X^{\varepsilon_{i+1}}T_{i+1}^{-1}$ and so

$$\begin{aligned} X^{\varepsilon_i} &= T_i^{-1}T_{i+1}^{-1}\cdots T_n^{-1}\sigma T_nT_{n-1}\cdots T_2T_1T_2\cdots T_i \\ &= \sigma T_i^{-1}T_{i+1}^{-1}\cdots T_{n-1}^{-1}T_0^{-1}T_nT_{n-1}\cdots T_2T_1T_2\cdots T_i. \end{aligned}$$

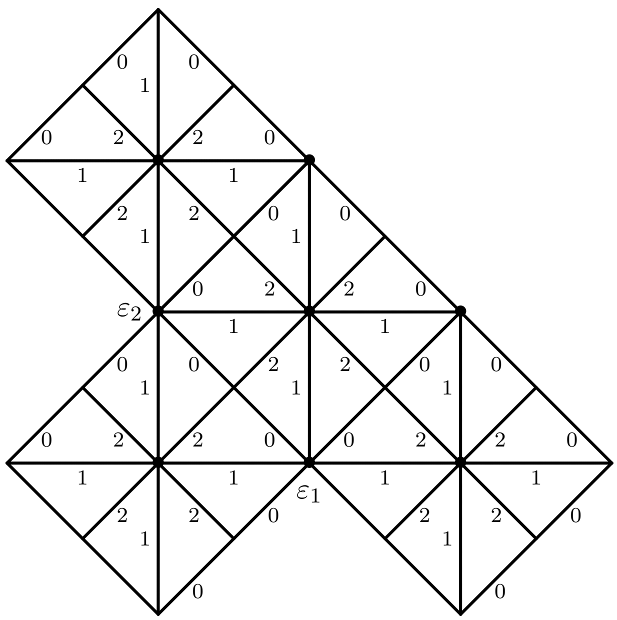
When $n = 2$: In this case the Dynkin diagram is 

and if

$g_2 =$ , $g_1 =$ , $g_0 =$ 

then

$$X^{\varepsilon_2} = \sigma T_2 T_1 T_2 = T_0 T_1 T_0 \sigma = PICTURE,$$
$$X^{\varepsilon_1} = T_0 T_2^{-1} T_1 \sigma = \sigma T_2 T_0^{-1} T_1 = PICTURE,$$
$$X^{\varepsilon_1+\varepsilon_2} = T_0 T_1 T_2 T_1 = PICTURE.$$

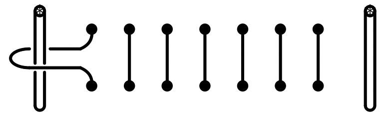


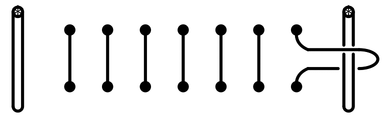
The *affine braid group of type C* $\tilde{\mathcal{B}}$ is given by generators $g_0, \ldots, g_n$ and relations according to the Dynkin diagram of type C

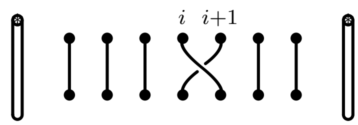
so that

$$g_1 g_2 g_1 g_2 = g_2 g_1 g_2 g_1, \qquad \text{and} \qquad g_0 g_n g_0 g_n = g_n g_0 g_n g_0,$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad \text{for } 2 \le i \le n-1, \quad \text{and} \quad g_i g_j = g_j g_i, \quad \text{for } |i-j| > 1,$$
$$g_0 g_i = g_i g_0, \quad \text{for } 1 \le i \le n-1.$$

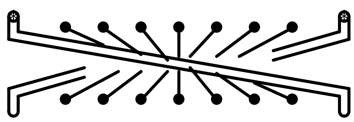
A pictorial representation $\tilde{\mathcal{B}}$ is

$g_1 =$ 

$g_0 =$  and

$g_i =$ 

It may be helpful to add to $\tilde{\mathcal{B}}$ the full twist

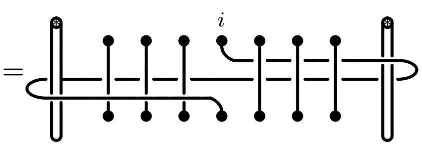
$\sigma =$  so that $\sigma g_1 \sigma^{-1} = g_0$, and $\sigma g_i \sigma^{-1} = g_{n-i+2}$,

produces the automorphism of the Dynkin diagram.

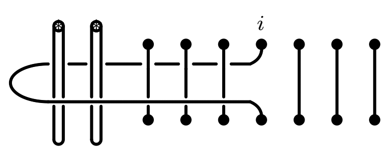
This pictorial representation indicates that there are R-matrix representations of $\tilde{\mathcal{B}}$ as follows. Let U be a quasitriangular Hopf algebra. Let M_1, M_2 and V be U-modules. Then the map

$$\begin{array}{ccc} \mathbb{C}\tilde{\mathcal{B}} & \longrightarrow & \mathrm{End}_u(M_1 \otimes V^{\otimes k} \otimes M_2) \\ g_1 & \longmapsto & \check{R}_{M_1,V}\check{R}_{V,M_1} \otimes \mathrm{id}_V^{\otimes(k-1)} \otimes \mathrm{id}_{M_2} \\ g_i & \longmapsto & \mathrm{id}_{M_1} \otimes \mathrm{id}_V^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}_V^{\otimes(k-i-1)} \otimes \mathrm{id}_{M_2} \\ g_0 & \longmapsto & \mathrm{id}_{M_1} \otimes \mathrm{id}_V^{\otimes(k-1)} \otimes \check{R}_{V,M_2}\check{R}_{M_2,V}. \end{array}$$

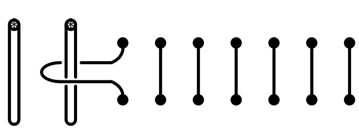
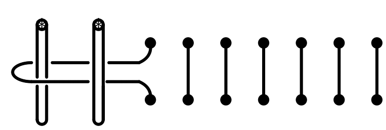
Then

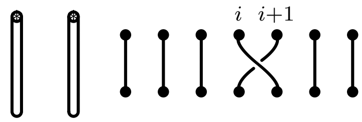
$X^{\varepsilon_i} =$ 

There is an isomorphism moving the right hand pole to the left, after which

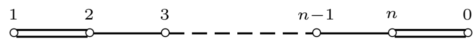
$X^{\varepsilon_i} =$ 

In this new notation

$g_1 =$  $g_0 =$ 

and $\quad g_i =$ 

The Dynkin diagram of affine type C is



Then

$$\begin{aligned} \alpha_1 &= 2\varepsilon_1, & \alpha_1^\vee &= \varepsilon_1, & \omega_1^\vee &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n), \\ \alpha_i &= \varepsilon_i - \varepsilon_{i-1}, & \alpha_i^\vee &= \varepsilon_i - \varepsilon_{i-1}, & \omega_i^\vee &= \varepsilon_i + \cdots + \varepsilon_n, \end{aligned}$$

$$P^\vee = \{\lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \text{all } \lambda_i \in \tfrac{1}{2}\mathbb{Z}_{\geq 0} \text{ or all } \lambda_i \in \mathbb{Z}_{\geq 0}\}, \qquad Q^\vee = \sum_{i=1}^n \mathbb{Z}\varepsilon_i.$$

Then

$$\varphi = 2\varepsilon_n, \quad \varphi^\vee = \varepsilon_n, \qquad \text{and} \qquad s_\varphi = t_n = s_n s_{n-1} \cdots s_2 s_1 s_2 \cdots s_n.$$

So

$$X^{\varepsilon_n} = T_0 T_n T_{n-1} \cdots T_2 T_1 T_2 \cdots T_n.$$

Then, since $X^{\varepsilon_i} = T_{i+1}^{-1} X^{\varepsilon_{i+1}} T_{i+1}^{-1}$,

$$X^{\varepsilon_i} = T_{i+1}^{-1} T_{i+2}^{-1} \cdots T_n^{-1} T_0 T_n T_{n-1} \cdots T_2 T_1 T_2 \cdots T_i.$$

Next

$$\omega_1^\vee = \frac{1}{2}(\varepsilon_1 + \cdots \varepsilon_n) \qquad \text{and} \qquad w_1 = s_{1n} s_{2,n-1} s_{3,n-3} \cdots, \qquad w_0 = t_1 t_2 \cdots t_n.$$

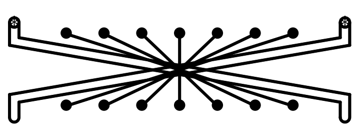
So

$$w_0 w_1 = (s_1 s_2 \cdots s_n)(s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1.$$

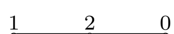
So

$$X^{\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)} = \sigma (T_1 T_2 \cdots T_n)(T_1 T_2 \cdots T_{n-1})(T_1 T_2 \cdots T_{n-2}) \cdots (T_1 T_2) T_1,$$

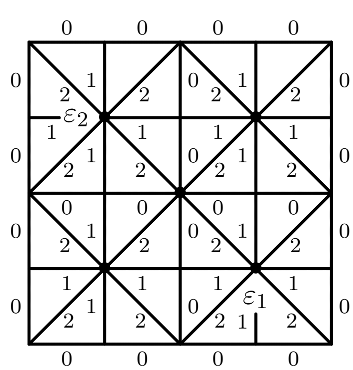
where

$\sigma =$ 

an element of order 2.

When $n = 2$: the Dynkin diagram is 

and the alcoves are



and

$$
\begin{aligned}
X^{\varepsilon_2} &= T_0T_2T_1T_2 = PICTURE, \\
X^{\varepsilon_1} &= T_2^{-1}T_0T_2T_1 = PICTURE, \\
X^{\frac{1}{2}(\varepsilon_1+\varepsilon_2)} &= \sigma T_1T_2T_1 = PICTURE.
\end{aligned}
$$

3 Affine type C Temperley-Lieb

Let $\tilde{H}$ be the quotient of $\mathbb{C}\tilde{B}$ by the relations

$$
\begin{gathered}
g_i^2 = (q-q^{-1})g_i + 1, \qquad \text{for } 1 \le i \le n-1, \\
g_0^2 = (s-s^{-1})g_0 + 1, \qquad \text{and} \qquad g_n^2 = (t-t^{-1})g_n + 1.
\end{gathered}
$$

Then let

$$
\begin{gathered}
e_i = q - g_i, \qquad \text{for } 1 \le i \le n-1, \\
e_0 = s - g_0, \qquad \text{and} \qquad e_n = t - g_n.
\end{gathered}
$$

Proposition 3.1.

(a) The relation

$$g_i^2 = (q-q^{-1})g_i + 1 \quad \textit{is equivalent to} \quad e_i^2 = (q_q^{-1})e_i.$$

(b) Assuming the relations $g_i^2 = (q-q^{-1})g_i + 1$*, the relation*

$$g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1} \quad \textit{is equivalent to} \quad e_ie_{i+1}e_i - e_{i+1}e_ie_{i+1} = e_i - e_{i+1}.$$

(c) Assuming the relations $g_i^2 = (q_i-q_i^{-1})g_i + 1$*, the relation*

$$g_0g_1g_0g_1 = g_1g_0g_1g_0 \quad \textit{is equivalent to} \quad e_0e_1e_0e_1 - e_1e_0e_1e_0 = (sq^{-1}+qs^{-1})(e_0e_1 - e_1e_0).$$

Define an algebra T_n generated by $e_0, e_1, \ldots, e_n$ with relations

$$
\begin{gathered}
e_1^2 = (s+s^{-1})e_1, \qquad e_i^2 = (q+q^{-1})^2, \qquad e_0^2 = (t+t^{-1})e_0, \\
e_2e_1e_2 = (sq^{-1}+qs^{-1})e_2, \qquad e_ie_{i-1}e_i = e_i, \qquad e_ie_{i+1}e_i = e_i, \qquad e_ne_0e_n = (tq^{-1}+qt^{-1})e_n.
\end{gathered}
$$

where $2 \le i \le n$. This algebra is a surjective image of $\tilde{H}$ with kernel generated by

???

Putting

$$I = \prod_{i \text{ even}} e_i \qquad \text{and} \qquad J = \prod_{i \text{ odd}} e_i$$

and imposing the relations

$$IJI = bI \qquad \text{and} \qquad JIJ = bJ$$

makes this into a finite dimensional algebra (see the work of Rittenberg, Nichols, de Gier and Pyatov).