The affine Weyl group

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1 The affine Weyl group

This section is a summary of the main facts and notations that are needed for working with the affine Weyl group $\tilde{W}$. The main point is that the elements of the affine Weyl group can be identified with alcoves via the bijection in (2.11).

Let $\mathfrak{h}_{\mathbb{R}}^*$ be a finite dimensional vector space over $\mathbb{R}$. A *reflection* is a diagonalizable element of $GL(\mathfrak{h}_{\mathbb{R}}^*)$ which has exactly one eigenvalue not equal to 1. A *lattice* is a free $\mathbb{Z}$-module. A *Weyl group* is a finite subgroup W of $GL(\mathfrak{h}_{\mathbb{R}}^*)$ which is

$$\text{generated by reflections} \qquad \text{and} \qquad \text{acts on a lattice } L \text{ in } \mathfrak{h}_{\mathbb{R}}^*$$

such that $\mathfrak{h}_{\mathbb{R}}^* = L \otimes_{\mathbb{Z}} \mathbb{R}$. Let R^+ be an index set for the reflections in W so that, for $\alpha \in R^+$,

$$s_\alpha \quad \text{is the reflection in the hyperplane} \quad H_\alpha = (\mathfrak{h}_{\mathbb{R}}^*)^{s_\alpha},$$

the fixed point space of the transformation s_α. The *chambers* are the connected components of the complement

$$\mathfrak{h}_{\mathbb{R}}^* \backslash \Big(\bigcup_{\alpha \in R^+} H_\alpha \Big)$$

of these hyperplanes in $\mathfrak{h}_{\mathbb{R}}^*$. These are fundamental regions for the action of W.

Let $\langle , \rangle$ be a nondegenerate W-invariant bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$. Fix a chamber C and choose vectors $\alpha^\vee \in \mathfrak{h}_{\mathbb{R}}^*$ such that

$$C = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha^\vee \rangle > 0\} \qquad \text{and} \qquad P \supseteq L \supseteq Q, \tag{1.1}$$

where

$$P = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\} \qquad \text{and} \qquad Q = \sum_{\alpha \in R^+} \mathbb{Z}\alpha, \quad \text{where} \quad \alpha = \frac{2\alpha^\vee}{\langle \alpha^\vee, \alpha^\vee \rangle}. \tag{1.2}$$

Pictorially,

$$\langle \lambda, \alpha^\vee \rangle \quad \text{is the distance from } \lambda \text{ to the hyperplane } H_\alpha.$$

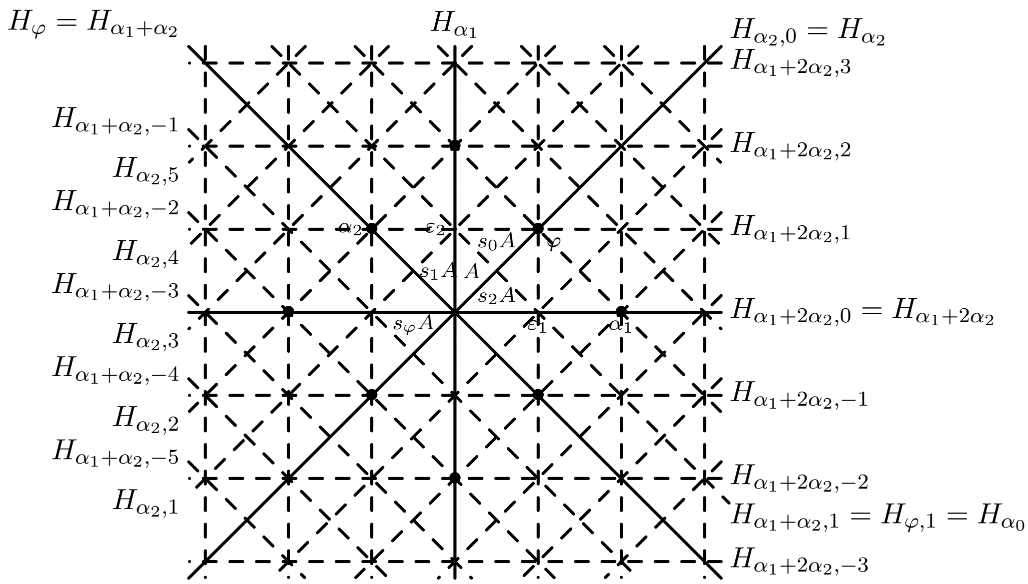
The *alcoves* are the connected components of the complement

$$\mathfrak{h}_{\mathbb{R}}^* \backslash \Big(\bigcup_{\substack{\alpha \in R^+ \\ j \in \mathbb{Z}}} H_{\alpha,j} \Big) \qquad \text{of the (affine) hyperplanes} \qquad H_{\alpha,j} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha^\vee \rangle = j\}$$

in $\mathfrak{h}_{\mathbb{R}}^*$. The *fundamental alcove* is the alcove

$$A \subseteq C \qquad \text{such that} \qquad 0 \in \overline{A}, \tag{1.3}$$

where $\overline{A}$ is the closure of A. An example is the case of type C_2, where the picture is



The *translation* in λ is the operator $t_\lambda \colon \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$ given by

$$t_\lambda(x) = \lambda + x, \qquad \text{for } \lambda \in P, x \in \mathfrak{h}_{\mathbb{R}}^*. \tag{1.4}$$

The reflection $s_{\alpha,k}$ in the hyperplane $H_{\alpha,k}$ is given by

$$s_{\alpha,k} = t_{k\alpha} s_\alpha = s_\alpha t_{-k\alpha}. \tag{1.5}$$

The *extended affine Weyl group* is

$$\widetilde{W} = P \rtimes W = \{t_\lambda w \mid \lambda \in P, w \in W\} \qquad \text{with} \qquad w t_\lambda = t_{w\lambda} w. \tag{1.6}$$

Denote the walls of C by $H_{\alpha_1}, \ldots, H_{\alpha_n}$ and extend this indexing so that

$$H_{\alpha_0}, \ldots, H_{\alpha_n} \qquad \text{are the walls of} \qquad A,$$

the fundamental alcove. Then the *affine Weyl group*,

$$W_{\mathrm{aff}} = Q \rtimes W \quad \text{is generated by } s_0, \ldots, s_n, \tag{1.7}$$

the reflections in the hyperplanes $H_{\alpha_0}, \ldots, H_{\alpha_n}$. Furthermore, A is a fundamental region for the action of W_{aff} on $\mathfrak{h}_{\mathbb{R}}^*$ and so there is a bijection

$$\begin{array}{ccc} W_{\mathrm{aff}} & \longrightarrow & \{\text{alcoves in } \mathfrak{h}_{\mathbb{R}}^*\} \\ w & \longmapsto & w^{-1}A. \end{array}$$

The *length* of $w \in \widetilde{W}$ is

$$\ell(w) = \text{number of hyperplanes between } A \text{ and } wA. \tag{1.8}$$

The difference between W_{aff} and $\widetilde{W}$ is the group

$$\Omega = \widetilde{W}/W_{\mathrm{aff}} \cong P/Q. \tag{1.9}$$

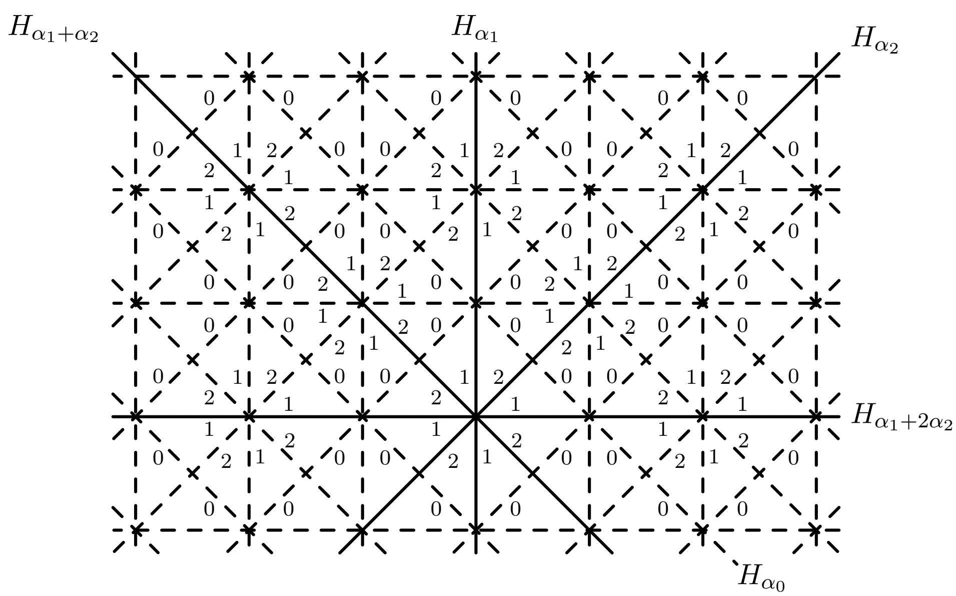
The group Ω is the set of elements of $\widetilde{W}$ of length 0. An element of Ω acts on the fundamental alcove A by an automorphism. Its action on A induces a permutation of the walls of A, and hence a permutation of $0, 1, \ldots, n$. If $g \in \Omega$ and $g \neq 1$ let ω_i be the image of the origin under the action of g on A. If s_j denotes the reflection in the jth wall of A and w_i denotes the longest element of the stabilizer W_{ω_i} of ω_i in W, then

$$g s_i g^{-1} = s_{g(i)} \qquad \text{and} \qquad g w_0 w_i = t_{\omega_i}. \tag{1.10}$$

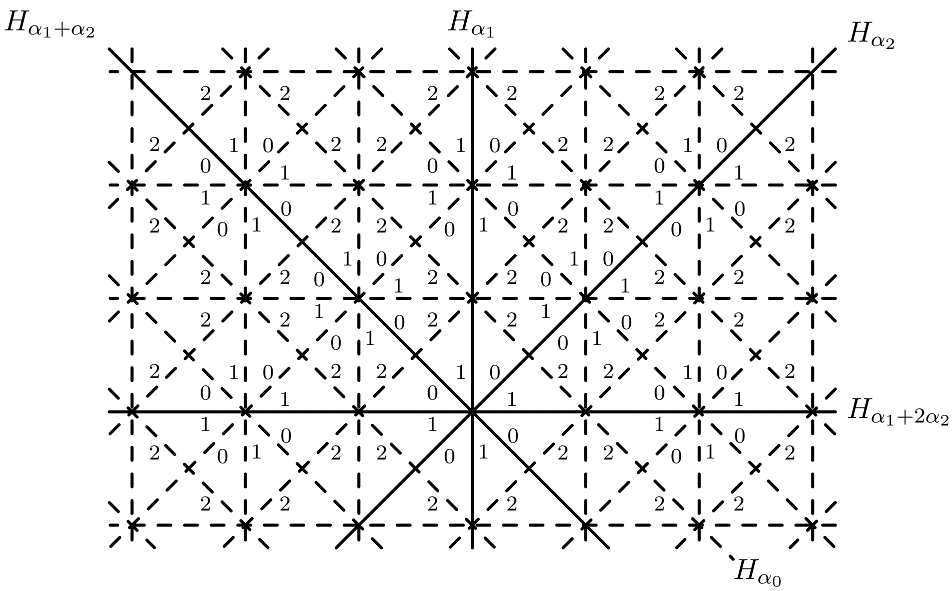
The group $\widetilde{W}$ acts freely on $\Omega \times \mathfrak{h}^*_{\mathbb{R}}$ ($|\Omega|$ copies of $\mathbb{R}^n$ tiled by alcoves) so that $g^{-1}A$ is in the same spot as A except on the gth "sheet" of $\Omega \times \mathfrak{h}^*_{\mathbb{R}}$. It is helpful to think of the elements of Ω as the *deck transformations* which transfer between the sheets in $\Omega \times \mathfrak{h}^*_{\mathbb{R}}$. Then

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \{\text{alcoves in } \Omega \times \mathfrak{h}^*_{\mathbb{R}}\} \\ w & \longmapsto & w^{-1}A \end{array} \tag{1.11}$$

is a bijection. In type C_2, the two sheets in $\Omega \times \mathfrak{h}^*_{\mathbb{R}}$ look like



and

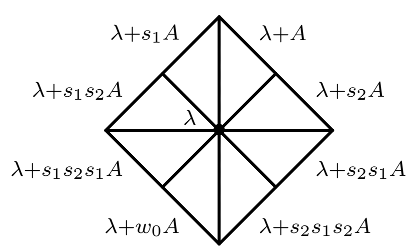


(1.12)

where the numbering on the walls of the alcoves is $\widetilde{W}$ equivariant so that, for $w \in \tilde{W}$, the numbering on the walls of wA is the w image of the numbering on the walls of A.

The *0-polygon* is the W-orbit of A in $\Omega \times \mathfrak{h}_{\mathbb{R}}^*$ and for $\lambda \in P$, the

the λ-*polygon* is $\lambda + WA$,



the translate of the W orbit of A by λ. The space $\Omega \times \mathfrak{h}_{\mathbb{R}}^*$ is tiled by the polygons and, via (2.11), we make identifications between W, $\widetilde{W}$, P and their geometric counterparts in $\Omega \times \mathfrak{h}_{\mathbb{R}}^*$:

$$\widetilde{W} = \{\text{alcoves}\}, \qquad W = \{\text{alcoves in the 0-polygon}\}, \qquad P = \{\text{centers of polygons}\}. \quad (1.13)$$

Define

$$P^+ = P \cap \overline{C} \qquad \text{and} \qquad P^{++} = P \cap C \quad (1.14)$$

so that P^+ is a set of representatives of the orbits of the action of W on P. The *fundamental weights* are the generators $\omega_1, \ldots, \omega_n$ of the $\mathbb{Z}_{\geq 0}$-module P^+ so that

$$C = \sum_{i=1}^n \mathbb{R}_{\geq 0}\omega_i, \qquad P^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i, \qquad \text{and} \qquad P^{++} = \sum_{i=1}^n \mathbb{Z}_{>0}\omega_i. \quad (1.15)$$

The lattice P has $\mathbb{Z}$-basis $\omega_1, \ldots, \omega_n$ and the map

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda, \end{array} \qquad \text{where} \quad \rho = \omega_1 + \ldots + \omega_n, \quad (1.16)$$

is a bijection. The *simple coroots* are $\alpha_1^\vee, \ldots, \alpha_n^\vee$ the dual basis to the fundamental weights,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}. \tag{1.17}$$

Define

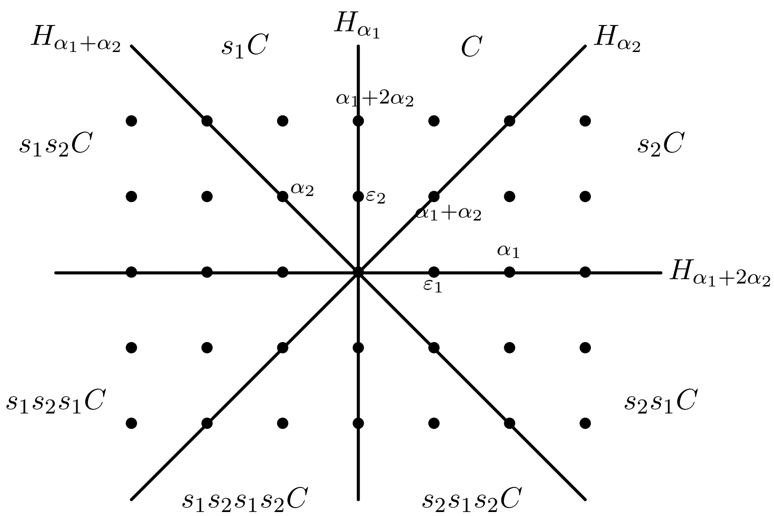
$$\overline{C^\vee} = \sum_{i=1}^{n} \mathbb{R}_{\le 0}\alpha_i^\vee \qquad \text{and} \qquad C^\vee = \sum_{i=1}^{n} \mathbb{R}_{<0}\alpha_i^\vee. \tag{1.18}$$

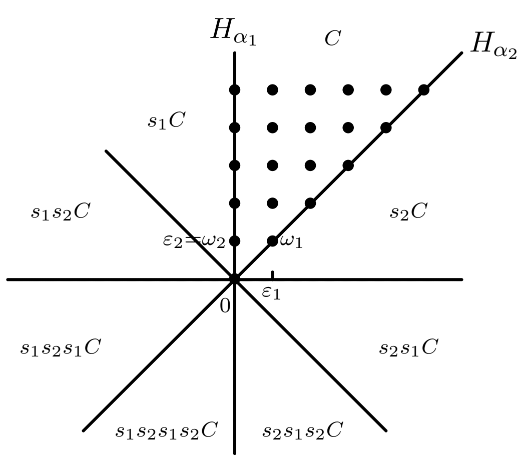
The *dominance order* is the partial order on $\mathfrak{h}_\mathbb{R}^*$ given by

$$\mu \le \lambda \qquad \text{if} \qquad \mu \in \lambda + \overline{C^\vee}. \tag{1.19}$$

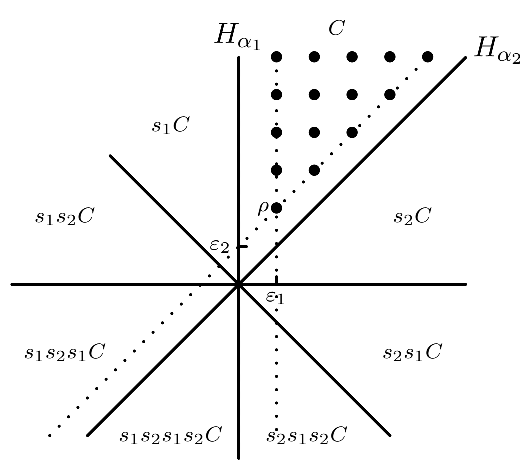
In type C_2 the lattice $P = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2$ with $\{\varepsilon_1, \varepsilon_2\}$ an orthonormal basis of $\mathfrak{h}_\mathbb{R}^* \cong \mathbb{R}^2$ and $W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$ is the dihedral group of order 8 generated by the reflections s_1 and s_2 in the hyperplanes H_{α_1} and H_{α_2}, respectively, where

$$H_{\alpha_1} = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \varepsilon_1 \rangle = 0\} \qquad \text{and} \qquad H_{\alpha_2} = \{x \in \mathfrak{h}_\mathbb{R}^* \mid \langle x, \varepsilon_2 - \varepsilon_1 \rangle = 0\}.$$





The set P^+



The set P^{++}

In this case

$$\begin{aligned} \omega_1 &= \varepsilon_1 + \varepsilon_2, & \alpha_1 &= 2\varepsilon_1, & \alpha_1^\vee &= \varepsilon_1, \\ \omega_2 &= \varepsilon_2, & \alpha_2 &= \varepsilon_2 - \varepsilon_1, & \alpha_2^\vee &= \alpha_2, \end{aligned}$$

and

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\}.$$