Abstract crystals

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1 Abstract Crystals

Let $C = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i \in I}$ be a Cartan matrix. Define free abelian groups

$$P = \sum_{i \in I} \mathbb{Z}\omega_i \qquad \text{and} \qquad Q = \sum_{i \in I} \mathbb{Z}\alpha_i, \qquad \text{and a pairing} \qquad \langle , \rangle \colon P \times Q \to \mathbb{Z}$$

given by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

A *crystal* is a set B with maps

$$\mathrm{wt} \colon B \to P$$

$$\varepsilon_i \colon B \to \mathbb{Z} \cup \{-\infty\} \qquad \text{and} \qquad \varphi_i \colon B \to \mathbb{Z} \cup \{-\infty\},$$

$$\tilde{e}_i \colon B \to B \cup \{0\} \qquad \text{and} \qquad \tilde{f}_i \colon B \to B \cup \{0\},$$

such that

(1) If $\tilde{e}_i b \neq 0$ then

$$\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i, \quad \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \quad \tilde{f}_i \tilde{e}_i b = b,$$

and if $\tilde{f}_i b \neq 0$ the

$$\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i, \quad \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \quad \tilde{e}_i \tilde{f}_i b = b,$$

(2) $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$, and

(3) If $\varphi_i(b) = -\infty$ then $\tilde{e}_i b = \tilde{f}_i b = 0$.

The *crystal graph* of B is the graph with

$$\text{vertex set } B \qquad \text{and} \qquad \text{labeled edges} \quad b \overset{i}{\longleftrightarrow} \tilde{e}_i b \quad \text{when } \tilde{e}_i b \neq 0.$$

The μ-weight space of a crystal B is the set

$$B_\mu = \{ b \in B \mid \mathrm{wt}(b) = \mu \}.$$

The *character* of B is the weight generating function of B,

$$\chi^B = \sum_{b\in B} X^{\mathrm{wt}(b)} = \sum_{\mu\in P} \mathrm{Card}(B_\mu) X^\mu \qquad \in \mathbb{Z}[P].$$

A *normal crystal* is a crystal B such that

$$\varepsilon_i(b) = \max\{k \mid \tilde e_i^k b \neq 0\} \qquad \text{and} \qquad \varphi_i(b) = \max\{k \mid \tilde f_i^k b \neq 0\}.$$

If B is a normal crystal and $b\in B$ the *i-string* of b is the set

$$\tilde f_i^{\varphi_i(b)} b \stackrel{i}{\longleftrightarrow} \cdots \stackrel{i}{\longleftrightarrow} \tilde f_i^2 b \stackrel{i}{\longleftrightarrow} \tilde f_i b \stackrel{i}{\longleftrightarrow} b \stackrel{i}{\longleftrightarrow} \tilde e_i b \stackrel{i}{\longleftrightarrow} \tilde e_i^2 b \stackrel{i}{\longleftrightarrow} \cdots \stackrel{i}{\longleftrightarrow} \tilde e_i^{\varepsilon_i(b)} b\},$$

and (3) is equivalent to $\langle \mathrm{wt}(\tilde e_i^{\varepsilon_i(b)} b), \alpha_i^\vee\rangle = -\langle \mathrm{wt}(\tilde f_i^{\varphi(b)} b), \alpha_i^\vee\rangle$ so that *every i string in a normal crystal B is a model for a finite dimensional $\mathfrak{sl}_2$-module.*

If B is a normal crystal define a bijection $s_i\colon B\to B$ by

$$s_i b = \begin{cases} \tilde f_i^{\mathrm{wt}_i(b)} b, & \text{if } \mathrm{wt}_i(b)\ge 0,\\ \tilde e_i^{-\mathrm{wt}_i(b)} b, & \text{if } \mathrm{wt}_i(b)\le 0,\end{cases} \qquad \text{so that} \qquad \mathrm{wt}(s_i b) = s_i \mathrm{wt}(b), \qquad \text{for all } b\in B.$$

The map s_i flips each i-string in B. The equality $\mathrm{wt}(s_i b) = s_i\mathrm{wt}(b)$ implies

$$\chi^B \in \mathbb{Z}[P]^W, \qquad \text{for any normal crystal } B.$$

Proposition 1.1 (Kashiwara, Duke 73 (1994), 383-413). *Let B be a normal crystal. The maps $s_i\colon B\to B$ $i\in I$, define an action of W on B.*

Proof. $\square$

Let B_1 and B_2 be crystals. A *morphism* $\psi\in\mathrm{Hom}(B_1,B_2)$ is a map $\phi\colon B_1\to B_2\cup\{0\}$ such that

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \qquad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \qquad \varphi_i(\psi(b)) = \varphi_i(b),$$

and

$$\text{if } b \stackrel{i}{\longleftrightarrow} \tilde e_i b \text{ and } \psi(b)\neq 0 \text{ and } \psi(\tilde e_i b)\neq 0 \qquad \text{then} \qquad \psi(b) \stackrel{i}{\longleftrightarrow} \tilde e_i\psi(b)$$

A *strict morphism* is a morphism that commutes with all $\tilde e_i$ and all $\tilde f_i$.

The *tensor product* of B_1 and B_2 is the crystal

$$B_1\otimes B_2 = \{b_1\otimes b_2 \mid b_1\in B_1, b_2\in B_2\} \qquad \text{with}$$

$$\mathrm{wt}_i(b_1\otimes b_2) = \mathrm{wt}_i(b_1)+\mathrm{wt}_i(b_2), \qquad \begin{array}{rl} \varepsilon_i(b_1\otimes b_2) & = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1),\alpha_i^\vee\rangle\},\\ \varphi_i(b_1\otimes b_2) & = \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2),\alpha_i^\vee\rangle, \varphi_i(b_2)\},\end{array}$$

$$\tilde e_i(b_1\otimes b_2) = \begin{cases} \tilde e_i b_1\otimes b_2, & \text{if } \varphi_i(b_1)\ge\varepsilon_i(b_2),\\ b_1\otimes\tilde e_i b_2, & \text{if } \varphi_i(b_1)<\varepsilon_i(b_2),\end{cases} \quad \text{and} \quad \tilde f_i(b_1\otimes b_2) = \begin{cases} \tilde f_i b_1\otimes b_2, & \text{if } \varphi_i(b_1)>\varepsilon_i(b_2),\\ b_1\otimes\tilde f_i b_2, & \text{if } \varphi_i(b_1)\le\varepsilon_i(b_2),\end{cases}$$

If B_1, B_2, B_3 are crystals, then the map

$$\begin{array}{ccc} (B_1\otimes B_2)\otimes B_3 & \stackrel{\sim}{\longrightarrow} & B_1\otimes(B_2\otimes B_3)\\ (b_1\otimes b_2)\otimes b_3 & \longmapsto & b_1\otimes(b_2\otimes b_3)\end{array}$$

is a crystal isomorphism and so we may simply write $B_1\otimes B_2\otimes B_3$ for the tensor product of B_1, B_2 and B_3.

Lemma 1.2. *If B_1 and B_2 are normal crystals then $B_1 \otimes B_2$ is normal.*

Proof. $\square$

If B is a crystal the *dual crystal* is the crystal $B^*\{b^* \mid b \in B\}$ with $\mathrm{wt}(b^*) = -\mathrm{wt}(b)$,

$$\varepsilon_i(b^*) = \phi_i(b), \qquad \text{and} \quad \varphi(b^*) = \varepsilon_i(b),$$

$$\tilde{e}_i(b^*) = (\tilde{f}_i b)^*, \qquad \text{and} \quad \tilde{f}_i(b^*) = (\tilde{e}_i b)^*.$$

The crystal graph of B^* is obtained by reversing all the arrows in the crystal graph of B.

1.1 Irreducible crystals $B(\lambda)$

A normal crystal B is *irreducible* if the crystal graph of B has a single connected component??? A *highest weight path* is an element $b \in B$ such that $\tilde{e}_i b = 0$ for all $i \in I$.

Theorem 1.3. *The irreducible highest weight crystals $B(\lambda)$ are indexed by $\lambda \in P^+$.*

Proof. $\square$

We would like to show that there is a unique normal crystal $B(\lambda)$ of highest weight λ. Define

$$B(\lambda+\mu) \text{ is the connected component of } b_\lambda^+ \otimes b_\mu^+ \text{ in } B(\lambda) \otimes B(\mu).$$

Thus, by definition there is a canonical injection

$$\begin{array}{rccc} \iota_{\lambda+\mu}^{\lambda\otimes\mu}\colon & B(\lambda+\mu) & \hookrightarrow & B(\lambda)\otimes B(\mu) \\ & b_{\lambda+\mu}^+ & \longmapsto & b_\lambda^+ \otimes b_\mu^+ \end{array}$$

Proposition 1.4. *Let $\lambda + \mu \in P^+$. The crystal $B(\lambda+\mu)$ is well defined, i.e*

$$B(\lambda+\mu) \cong B(\gamma+\delta) \quad \textit{if} \quad \lambda+\mu = \gamma+\delta.$$

Proof. $\square$

This reduces the problem of finding $B(\lambda)$ to the fundamental weights.

Theorem 1.5. *Let $\lambda \in P^+$. Then $B(\lambda)$ exists.*

Another characterization?

$$B(\lambda) = \{b \otimes b_\lambda^{-\infty} \in B(\infty) \otimes B_\lambda^{-\infty} \mid \varepsilon_i^*(b) \le \langle \lambda, \alpha_i^\vee \rangle, \text{ for all } i \in I\}.$$

For each $m \in \mathbb{Z}_{>0}$ and each $\lambda \in P^+$ there unique injective maps

$$\begin{array}{rccc} S_m\colon & B(\lambda) & \longrightarrow & B(m\lambda) \\ & b_\lambda^+ & \longmapsto & b_{m\lambda}^+ \end{array}$$

such that

$$\mathrm{wt}(S_m b) = m\mathrm{wt}(b), \qquad \varepsilon_i(S_m b) = m\varepsilon_i(b), \qquad \varphi(S_m b) = m\varphi_i(m), \qquad \text{and}$$

$$S_m(\tilde{e}_i b) = \tilde{e}_i^m S_m(b) \qquad \text{and} \qquad S_m(\tilde{f}_i b) = \tilde{f}_i^m S_m(b) b.$$

Proposition 1.6.

$$B(\lambda)^\vee = B(-w_0\lambda).$$

1.2 The crystal $B(\infty)$

Define projections $\pi_\lambda^{\lambda+\mu}\colon B(\lambda+\mu) \to B(\lambda)$ by the composition

$$\begin{array}{rcccll} \pi_\lambda^{\lambda+\mu}\colon B(\lambda+\mu) & \hookrightarrow & B(\lambda)\otimes B(\mu) & \longrightarrow & B(\lambda) & \\ & & b\otimes b_\mu^+ & \longmapsto & b, & \\ & & b\otimes b' & \longmapsto & 0, & \qquad \text{if } b' \neq b_\mu^+. \end{array}$$

The projective system defined by the $\pi_\lambda^{\lambda+\mu}$ allows us to define

$$B(\infty) = \varprojlim B(\lambda) \qquad \text{so that} \qquad \pi_\lambda\colon B(\infty) \to B(\lambda), \quad \text{is such that} \quad \pi_\lambda^{\lambda+\mu}\pi_{\lambda+\mu} = \pi_\lambda,$$

for all $\lambda \in P^+$.

For each $j \in I$ define a crystal

$$B_j(\mathbb{Z}) = \{b_j(n) \mid n \in \mathbb{Z}\} \qquad\qquad \cdots \xleftrightarrow{j} b_j(-1) \xleftrightarrow{j} b_j(0) \xleftrightarrow{j} b_j(1) \xleftrightarrow{j} \cdots$$

with

$$\mathrm{wt}(b_j(n)) = n\alpha_j, \qquad \varepsilon_i(b_j(n)) = \begin{cases} -n, & \text{if } i=j, \\ -\infty, & \text{if } i\neq j, \end{cases} \qquad \varphi_i(b_j(n)) = \begin{cases} n, & \text{if } i=j, \\ -\infty, & \text{if } i\neq j, \end{cases}$$

and

$$\tilde{e}_i(b_j(n)) = \begin{cases} b_j(n+1), & \text{if } i=j, \\ 0, & \text{if } i\neq j, \end{cases} \qquad \text{and} \qquad \tilde{f}_i(b_j(n)) = \begin{cases} b_j(n-1), & \text{if } i=j, \\ 0, & \text{if } i\neq j, \end{cases}$$

Theorem 1.7.

(a) For each $j \in I$ there is a crystal injection

$$\begin{array}{rcccl} \Phi_j\colon & B(\infty) & \longrightarrow & B(\infty)\otimes B_j(\mathbb{Z}) & \\ & b_0 & \longmapsto & b_0 \otimes b_j(0) & \\ & b & \longmapsto & b' \otimes \tilde{f}_j^n b_j(0), & \qquad \text{with } n>0 \text{ if } b\neq b_0. \end{array}$$

(b) Let $(i_1, i_2, \ldots)$ be a sequence of elements of I such that each $i \in I$ appears an infinite number of times. Then the subcrystal of $\cdots \otimes B_{i_2}(\mathbb{Z}) \otimes B_{i_1}(\mathbb{Z})$ given by

$$B(\infty) = \{\cdots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) \mid a_i \in \mathbb{Z}_{\geq 0},\ a_k = 0 \textit{ for } k >> 0\},$$

the subcrystal of $\cdots \otimes B_{i_2}(\mathbb{Z}) \otimes B_{i_1}(\mathbb{Z})$ generated by $\cdots b_{i_2}(0) \otimes b_{i_1}(0)$.

If $\{i_1, i_2, \cdots\}$ is a sequence of elements of I such that each $i \in I$ appears an infinite number of times the composition $\cdots \Phi_{i_3} \circ \Phi_{i_2} \circ \Phi_{i_1}$ realize

$$B(\infty) = \{\cdots \otimes b_{i_2}(-a_2) \otimes b_{i_1}(-a_1) \mid a_i \in \mathbb{Z}_{\geq 0},\ a_k = 0 \text{ for } k >> 0\}.$$

For each $\lambda \in P^+$, define a crystal $B_\lambda^{-\infty} = \{b_\lambda^{-\infty}\}$ with

$$\mathrm{wt}(b_\lambda^{-\infty}) = \lambda, \quad \varepsilon_i(b_\lambda^{-\infty}) = -\infty, \quad \varphi_i(b_\lambda^{-\infty}) = -\infty, \quad \tilde{e}_i(b_\lambda^{-\infty}) = 0, \quad \tilde{f}_i(b_\lambda^{-\infty}) = 0,$$

for all $i \in I$. Then

$$B(\lambda) \text{ is the normal subcrystal of } B(\infty)\otimes B_\lambda^{-\infty} \text{ generated by } b_0^+ \otimes b_\lambda^{-\infty}.$$

Define $*\colon B(\infty) \to B(\infty)$ to be the unique involution such that

$$\Phi_i((\tilde{f}_i^n b^*)^*) = b \otimes \tilde{f}_i^n b_i(0), \qquad \text{for all } i \in I.$$

THIS DEFINITION NEEDS REWORKING!

Theorem 1.8. Kashiwara-Saito, Duke Math. J. **89** No. 1 (1997) *The set $B(\infty)$ endowed with the maps $\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i$ and $\tilde{f}_i$ is a crystal isomorphic to the crystal base of $U_q^-\mathfrak{g}$.*

1.3 Representation crystals

The *quantum group* is the $\mathbb{Q}(q)$ algebra given by generators

$$E_i, \quad F_i, \quad K_i, \quad K_i^{-1}, \qquad i \in I,$$

with relations

$$E_iF_j - F_jE_i = \delta_{ij}\left(\frac{K_i - K_i^{-1}}{q - q^{-1}}\right)$$

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad\qquad K_iK_j = K_jK_i,$$

$$K_iE_j = q^{\langle \alpha_j, \alpha_i^\vee\rangle} E_jK_i, \qquad\qquad K_iF_j = q^{-\langle \alpha_j, \alpha_i^\vee\rangle} F_jK_i,$$

$$0 = \sum_{r=0}^{\ell_{ij}} \begin{bmatrix} \ell_{ij} \\ r \end{bmatrix} E_i^r E_j E_i^{\ell_{ij}-r}, \qquad 0 = \sum_{r=0}^{\ell_{ij}} \begin{bmatrix} \ell_{ij} \\ r \end{bmatrix} F_i^r F_j F_i^{\ell_{ij}-r}, \qquad \text{for } i \neq j,$$

where $\ell_{ij} = -\langle \alpha_j, \alpha_i^\vee\rangle + 1$, and

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \qquad [k]! = [k][k-1]\cdots[2][1], \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Theorem 1.9. (Drinfel'd) *The algebra $U_q\mathfrak{g}$ is the unique Cartan preserving Hopf algebra deformation of G.*

Proof. $\square$

An *integrable $U_q\mathfrak{g}$-module* is a $U_q\mathfrak{g}$-module M such that

$$M = \bigoplus_{\mu \in P} M_\mu, \qquad \text{where} \qquad M_\mu = \{m \in M \mid K_i m = q^{\langle \mu, \alpha_i^\vee\rangle} m, \text{ for all } i \in I\},$$

and for each $m \in M$ and $i \in I$, $E_i^k m = 0$ and $F_i^k m = 0$ for $k >> 0$.

Theorem 1.10. *There is a bijection*

$$\begin{matrix} \{\textit{simple integrable } U_q\mathfrak{g}\textit{-modules}\} & \overset{1-1}{\longleftrightarrow} & P^+ \\ L(\lambda) & \leftrightarrow & \lambda \end{matrix}$$

Proof. $\square$

Let M be a integrable $U_q\mathfrak{g}$-module. The *crystal operators* $\tilde{e}_i \colon M \to M$ and $\tilde{f}_i \colon M \to M$ are the linear operators determined by

$$\tilde{e}_i(F_i^{(k)} m) = F_i^{(k-1)} m \qquad \text{and} \qquad \tilde{f}_i(F_i^{(k)} m) = F_i^{(k+1)} m,$$

for all $k \in \mathbb{Z}_{\geq 0}$ and $m \in M$ such that $E_i m = 0$ and $F_i^{(k)} m \neq 0$. The convention is that $F_i^{(-1)} m = 0$.

A *crystal basis* of M is a pair (L, B),

$$L = \bigoplus_{\mu \in P} L_\mu, \qquad B = \bigsqcup_{\mu \in P} B_\mu, \qquad \text{where} \qquad \begin{array}{l} L_\mu \text{ is a free } \mathbb{Q}[q]\text{-module with} \\ M_\mu = \mathbb{Q}(q) \otimes_{\mathbb{Q}[q]} L_\mu, \\ B_\mu \text{ is a basis of } L_\mu/qL_\mu, \end{array}$$

for all $\mu \in P$, and such that L is stable under $\tilde{e}_i$ and $\tilde{f}_i$ and the images of the operators $\tilde{e}_i$ and $\tilde{f}_i$ on L_μ/qL_μ with the definitions

$$\mathrm{wt}(b) = \mu, \qquad \text{if } b \in B_\mu,$$
$$\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq 0\}, \qquad \text{and} \qquad \varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq 0\},$$

make B into a crystal.

For each $\lambda \in P^+$ let $L(\lambda)$ be the irreducible $U_q\mathfrak{g}$-module of highest weight λ and fix a highest weight vector v_λ^+ in $L(\lambda)$. Define homomorphisms of $U_q^-\mathfrak{g}$-modules

$$\begin{array}{rccc} \pi_\lambda\colon & U_q^-\mathfrak{g} & \longrightarrow & L(\lambda) \\ & u & \longmapsto & uv_\lambda^+ \end{array}$$

Define

$$\mathcal{L}(\lambda) = \mathbb{Q}[q]\text{-span}\{\tilde{f}_{i_k}\cdots\tilde{f}_{i_1}v_\lambda^+ \mid i_1,\ldots,i_k \in I\}, \qquad \text{and}$$
$$B(\lambda) = \{\text{images of } \tilde{f}_{i_k}\cdots\tilde{f}_{i_1}v_\lambda^+ \text{ in } \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)\}.$$

Theorem 1.11.

(a) Let $\lambda \in P^+$ and let $L(\lambda)$ be the irreducible $U_q\mathfrak{g}$ module of highest weight λ. Then $(\mathcal{L}(\lambda), B(\lambda))$ is a crystal basis of $L(\lambda)$.

(b) There is a unique crystal basis $(\mathcal{L}(\infty), B(\infty))$ of $U_q^-\mathfrak{g}$ such that, for all $\lambda \in P^+$,

$$\pi_\lambda(\mathcal{L}(\infty)) = \mathcal{L}(\lambda) \qquad \textit{and} \qquad \bar{\pi}_\lambda(B(\infty)) = B(\lambda) \cup \{0\},$$

where $\bar{\pi}_\lambda\colon \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ is the map induced by π_λ.

1.4 Quiver crystals

Let $(I, \Omega^\pm)$ be the directed graph with vertex set I and an edge $(i \to j) \in \Omega^\pm$ if $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. Fix an *orientiation* of $(I, \Omega^\pm)$, i.e. a map

$$\begin{array}{rccc} c\colon & \Omega^\pm & \longrightarrow & \mathbb{C}^* \\ & i{\to}j & \longmapsto & c_{i\to j} \end{array} \qquad \text{such that} \qquad c_{i\to j} + c_{j\to i} = 0.$$

Recall that

$$P^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\omega_i \qquad \text{and} \qquad Q^- = -\sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i.$$

Fix

$$\lambda = \sum_{i\in I} \lambda_i\omega_i \in P^+ \qquad \text{and the } I\text{-graded vector space} \qquad \mathbb{C}^\lambda = \bigoplus_{i\in I} \mathbb{C}^{\lambda_i}.$$

For each

$$-\nu = \sum_{i\in I} -\nu_i\alpha_i \in Q^- \qquad \text{fix the } I\text{-graded vector space} \qquad \mathbb{C}^\nu = \bigoplus_{i\in I} \mathbb{C}^{\nu_i}.$$

There is a natural $GL_\nu = \prod_{i\in I} GL_{\nu_i}(\mathbb{C})$ action on the variety

$$\begin{aligned} X(\lambda)_{\lambda-\nu} &= \left(\bigoplus_{(i\to j)\in\Omega^\pm} \mathrm{Hom}(\mathbb{C}^{\nu_i}, \mathbb{C}^{\nu_j})\right) \oplus \left(\bigoplus_{i\in I} \mathrm{Hom}(\mathbb{C}^{\nu_i}, \mathbb{C}^{\lambda_i})\right) \oplus \left(\bigoplus_{i\in I} \mathrm{Hom}(\mathbb{C}^{\lambda_i}, \mathbb{C}^{\nu_i})\right) \\ &= \left\{ x = \left(\bigoplus_{(i\to j)\in\Omega^\pm} x_{i\to j}\right) \oplus \left(\bigoplus_{i\in I} x_{i\to}\right) \oplus \left(\bigoplus_{i\in I} x_{i\leftarrow}\right) \;\middle|\; \begin{array}{l} x_{i\to j} \in \mathrm{Hom}(\mathbb{C}^{\nu_i}, \mathbb{C}^{\nu_j}), \\ x_{i\to} \in \mathrm{Hom}(\mathbb{C}^{\nu_i}, \mathbb{C}^{\lambda_i}), \\ x_{i\leftarrow} \in \mathrm{Hom}(\mathbb{C}^{\lambda_i}, \mathbb{C}^{\nu_i}) \end{array} \right\} \end{aligned}$$

Write

$$x_{\leftrightarrow} = \bigoplus_{i\to j} x_{i\to j}\ , \qquad x_{\to} = \bigoplus_{i\in I} x_{i\to}\ , \quad \text{and} \quad x_{\leftarrow} = \bigoplus_{i\in I} x_{i\to}\ , \qquad \text{for } x\in X(\lambda)_{\lambda-\nu}.$$

Use the orientation to define a GL_ν-invariant symplectic form on $X(\lambda)_{\lambda-\nu}$ by

$$\omega(x,y) = \sum_{(i\to j)\in\Omega^\pm} c_{i\to j}\mathrm{Tr}(x_{j\to i}y_{i\to j}) + \sum_{i\in I}\mathrm{Tr}(x_{i\leftarrow}y_{i\to}) - \sum_{i\in I}\mathrm{Tr}(y_{i\to}x_{i\leftarrow}), \qquad \text{for } x,y\in X(\lambda)_{\lambda-\nu}.$$

The corresponding moment map

$$\mu\colon X(\lambda)_{\lambda-\nu}\to \mathfrak{gl}_\nu \qquad \text{is given by} \qquad \mu(x)_i = x_{i\leftarrow}x_{i\to} + \sum_{(i\to j)\in\Omega^\pm} c_{i\to j}(x_{j\to i}x_{i\to j}).$$

A point $x\in X(\lambda)_{\lambda-\nu}$ is *stable* if every $x_{\leftrightarrow}$-invariant I-graded vector space $U\subseteq \ker(x_{\mapsto})$ is 0. Let

$$\begin{aligned}
\mathcal{X}(\lambda)_{\lambda-\nu} &= \mu^{-1}(0)^{\mathrm{st}}/GL_\nu = \{\ GL_\nu\text{-orbits of stable points in } \mu^{-1}(0)\ \ \},\\
\Lambda(\lambda)_{\lambda-\nu} &= \{\ [x]\in\mathcal{X}(\lambda)_{\lambda-\nu}\mid x_{\mapsto}=0 \text{ and } x_{\leftrightarrow} \text{ is nilpotent}\ \},\\
B(\lambda)_{\lambda-\nu} &= \{\ \text{irreducible components of } \Lambda(\lambda)_{\lambda-\nu}\ \}.
\end{aligned}$$

If $b\in B(\lambda)_{\lambda-\nu}$ define

$$\varepsilon_i(b) = \varepsilon_i([x]) = \varepsilon_i(x) = \dim\Big(\mathrm{coker}\big(x_{i\leftarrow}\oplus\bigoplus_{j\in I} x_{i\leftarrow j}\big)\Big)$$

for a generic point $[x]$ in b. Let

$$X(\lambda)^{\lambda-\nu}_{\lambda-(\nu-d\alpha_i)} = \left\{ (x,V)\ \middle|\ \begin{array}{ll} x\in X(\lambda)_{\lambda-\nu}, & \mathrm{im}\,(x_{\leftarrow})\subseteq V\subseteq\mathbb{C}^\nu,\\ \dim(V)=\nu-d\alpha_i, & V \text{ is } x_{\leftrightarrow}\text{-stable}\end{array}\right\}$$

and define

$$\begin{array}{ccccc}
X(\lambda)_{\lambda-(\nu-d\alpha_i)} & \xleftarrow{q_1} & X(\lambda)^{\lambda-\nu}_{\lambda-(\nu-d\alpha_i)} & \xrightarrow{q_2} & X(\lambda)_{\lambda-\nu}\\
x\big|_V & \leftarrow & (x,V) & \mapsto & x
\end{array}$$

Passing to GL_ν-orbits of stable points these maps induce

$$\mathcal{X}(\lambda)^{\varepsilon_i=0}_{\lambda-(\nu-d\alpha_i)} \xleftarrow{q_1} q_2^{-1}\left(\mathcal{X}(\lambda)^{\varepsilon_i=d}_{\lambda-\nu}\right) \xrightarrow{q_2} \mathcal{X}(\lambda)^{\varepsilon_i=d}_{\lambda-\nu}$$

where $\mathcal{X}(\lambda)^{\varepsilon_i=d}_{\lambda-\nu} = \{\ [x]\in\mathcal{X}(\lambda)_{\lambda-\nu}\mid \varepsilon_i(x)=d\ \}$. The result is a bijection

$$\tilde{e}_i^d\colon B(\lambda)^{\varepsilon_i=0}_{\lambda-(\nu-d\alpha_i)} \xrightarrow{q_2\circ q_1^{-1}} B(\lambda)^{\varepsilon_i=d}_{\lambda-\nu}.$$

Theorem 1.12. *Let $\lambda\in P^+$. The set*

$$B(\lambda) = \bigsqcup_{\nu\in Q^-} B(\lambda)_{\lambda-\nu} \qquad \textit{with the maps} \qquad \tilde{e}_i\colon B(\lambda)\to B(\lambda)\cup\{0\}$$

defined by ??? is a realization of the highest weight crystal of weight λ.

Proof. □

Let (I, Ω^+) be a quiver, let Ω^- be the opposite orientation and let $\Omega^\pm = \Omega^+ \cup \Omega^-$. For each

$$-\nu = -\sum_{i \in I} \nu_i \alpha_i \quad \in Q^- \qquad \text{fix an } I\text{-graded vector space} \qquad V = \bigoplus_{i \in I} V_i,$$

such that $\dim(V_i) = \nu_i$. Let

$$\begin{array}{rl} G_V & = \prod_{i \in I} GL(V_i), \\ \mathfrak{gl}(V) & = \bigoplus_{i \in I} \mathrm{End}(V_i), \end{array} \qquad \text{and} \qquad \begin{array}{rl} E_V & = \bigoplus_{(i \to j) \in \Omega^+} \mathrm{Hom}(V_i, V_j), \\ X_V & = \bigoplus_{(i \to j) \in \Omega^\pm} \mathrm{Hom}(V_i, V_j). \end{array}$$

Define a moment map $\mu \colon X_V \to \mathfrak{gl}_V$ by

$$\mu = \oplus_{i \in I} \mu_i \qquad \text{where} \qquad \mu_i(x) = \sum_{(i \to j) \in \Omega^\pm} x_{j \to i} x_{i \to j} - x_{i \to j} x_{j \to i},$$

and let

$$\Lambda_V = \{x \in X_V \mid \mu(x) = 0 \ \text{ and } x \text{ is nilpotent}\}.$$

Let

$$B(\infty)_{-\nu} = \{\text{irreducible components of } \Lambda_V\} \qquad \text{and} \qquad B(\infty) = \bigsqcup_{-\nu \in Q^-} B(\infty)_{-\nu}.$$

For $b \in B$ define

$$\mathrm{wt}(b) = -\nu, \quad \text{if } b \in B(\infty)_{-\nu}, \qquad \text{and} \qquad \varepsilon_i(b) = \varepsilon_i(x) = \dim \mathrm{coker}\left(\left(\oplus_{j \to i} V_j \right) \xrightarrow{x_{j \to i}} V_i \right),$$

where x is any generic point of b.

Let $X_0(-\nu) = \{x \in X(-\nu) \mid \mu(x) = 0\}$ and let $X_0'(-\nu + r\alpha_i, -r\alpha_i)$ be the set of triples $(x, \bar{\phi}, \phi')$ such that

$$x \in X_0(-\nu), \quad \text{and} \quad 0 \to V(-\nu + r\alpha_i) \xrightarrow{\bar{\phi}} V(-\nu) \xrightarrow{\phi'} V(-r\alpha_i) \longrightarrow 0$$

is an exact sequence of I-graded vector spaces such that $\mathrm{im}\,(\bar{\phi})$ is x-stable. For each $(x, \bar{\phi}, \phi') \in X_0'(-\nu + r\alpha_i, -r\alpha_i)$ let

$$\bar{x} \colon V(-\nu + r\alpha_i) \longrightarrow V(-\nu + r\alpha_i) \qquad \text{and} \qquad x' \colon V(-r\alpha_i) \longrightarrow V(-r\alpha_i)$$

be the induced maps,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(\nu)_k & \xrightarrow{\bar{\phi}_k} & V(\nu)_k & \xrightarrow{\phi_k'} & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{x}_{k \to i}} & & \downarrow{\scriptstyle x_{k \to i}} & & \downarrow{\scriptstyle x'_{k \to i}} & & \\ 0 & \longrightarrow & W & \xrightarrow{\bar{\phi}_i} & V(\nu)_i & \xrightarrow{\phi_i'} & \mathbb{C}^r & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{x}_{i \to j}} & & \downarrow{\scriptstyle x_{i \to j}} & & \downarrow{\scriptstyle x'_{i \to j}} & & \\ 0 & \longrightarrow & V(\nu)_j & \xrightarrow{\bar{\phi}_j} & V(\nu)_j & \xrightarrow{\phi_j'} & 0 & \longrightarrow & 0 \end{array}$$

The maps

$$\begin{array}{ccccccc} X_0(-\nu + r\alpha_i) & \xleftarrow{\sim} & X_0(-\nu + r\alpha_i) \times X_0(-r\alpha_i) & \xleftarrow{\omega} & X_0'(-\nu + r\alpha_i, -r\alpha_i) & \xrightarrow{\pi_1} & X_0(-\nu) \\ \bar{x} & \longleftarrow & (\bar{x}, 0) = (\bar{x}, x') & \longleftarrow & (x, \bar{\phi}, \phi') & \longrightarrow & x \end{array}$$

induce an isomorphism

$$\tilde{e}_i^r = \omega \circ \pi_1^{-1} \colon \{b \in B(\infty)_{-\nu} \mid \varepsilon_i(b) = r\} \xrightarrow{\sim} \{b \in B(\infty)_{-\nu + r\alpha_i} \mid \varepsilon_i(b) = 0\}.$$

These maps determine maps $\tilde{e}_i$ on $B(\infty)$.

Theorem 1.13. *This is a realization of the crystal $B(\infty)$.*

Proof. □

1.5 Path crystals

Let $\vec{B}$ be the set of paths in $\mathfrak{h}_{\mathbb{R}}^*$ where a *path* in $\mathfrak{h}_{\mathbb{R}}^*$ is the image of a piecewise linear map

$$p\colon [0,1] \to \mathfrak{h}_{\mathbb{R}}^* \qquad \text{such that } p(0)=0 \text{ and } p(1) \in P.$$

PICTURE

Define functions

$$\mathrm{wt}\colon \vec{B} \to P,$$

$$\varepsilon_i(p)\colon \vec{B} \to \mathbb{Z}_{\geq 0} \qquad \text{and} \qquad \phi_i\colon \vec{B} \to \mathbb{Z}_{\geq 0}$$

$$\tilde{e}_i\colon \vec{B} \to \vec{B} \cup \{0\} \qquad \text{and} \qquad \tilde{f}_i\colon \vec{B} \to \vec{B} \cup \{0\}$$

by

$$\begin{aligned}
\mathrm{wt}(p) &= p(1) \\
&= \text{the endpoint of } p, \\
\varepsilon_i(p) &= \left| \lfloor \min\{\langle p(t), \alpha_i \vee \rangle \mid 0 \leq t \leq 1\} \rfloor \right| \\
&= \text{distance from the most negative point to } H_{\alpha_i}, \\
\varphi_i(p) &= \left| \lfloor \langle p(1), \alpha_i^\vee \rangle - \min\{\langle p(t), \alpha_i^\vee \rangle \mid 0 \leq t \leq 1\} \rfloor \right| \\
&= \text{distance from the most negative point to } p(1), \\
\tilde{e}_i(p) &= \begin{cases} t \mapsto p(t) + r_i(t)\alpha_i, & \text{if } r_i(0) = 0, \\ 0, & \text{otherwise,} \end{cases} \\
\tilde{f}_i(p) &= \begin{cases} t \mapsto p(t) - \ell_i(t)\alpha_i, & \text{if } \ell_i(1) = 1, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where $r_i\colon [0,1] \to [0,1]$ and $\ell_i\colon [0,1] \to [0,1]$ are the monotone functions given by

$$\begin{aligned}
r_i(t) &= 1 - \min\{1, \langle p(s), \alpha_i^\vee \rangle - \varepsilon_i(p) \mid 0 \leq s \leq t\} \\
\ell_i(t) &= \min\{1, \langle p(s), \alpha_i^\vee \rangle - \varepsilon_i(p) \mid t \leq s \leq 1\}
\end{aligned}$$

To visualize these operations note that

$$\langle p(t), \alpha_i^\vee \rangle \text{ is the “distance of the point } p(t) \text{ from the hyperplane } H_{\alpha_i}.\text{”}$$

PICTURE

For a path “traveling back and forth with respect to H_{α_i}”, place a hyperplane H_{mnp} parallel to H_{α_i} and through the most negative point of p. Draw another parallel hyperplane $H_{\mathrm{mnp}+1}$ one unit in the positive direction from H_{mnp} (i.e. $\langle x, \alpha_i^\vee \rangle = \langle y, \alpha_i^\vee \rangle + 1$ for $x \in H_{\mathrm{mnp}}$, $y \in H_{\mathrm{mnp}+1}$). Water poured down the tube created by H_{mnp} and $H_{\mathrm{mnp}+1}$ will create a waterfall and wet those parts of p corresponding to where the function ℓ_i is increasing.

PICTURE

The new path $\tilde{f}_i p$ is the path which follows the same trajectory as p except that the “wet parts” are “reflected with respect to the α_i direction”. In the case, figure 2, where the “the water flows in the positive direction” then $\tilde{f}_i p = 0$.

Proposition 1.14. *The set $\vec{B}$ is a normal crystal.*

Proof. □

Define the *concatenation* or *tensor product* of paths p_1 and p_2 to be the path $p_1 \otimes p_2$ given by

$$(p_1 \otimes p_2)(t) = \begin{cases} p_1(2t), & 0 \le t \le 1/2, \\ p_1(1) + p_2(2t-1), & 1/2 \le t \le 1. \end{cases}$$

The *reverse* of the path p is the path p^* given by

$$p^*(t) = p(1-t), \qquad 0 \le t \le 1.$$

For $k \in \mathbb{Z}_{\ge 0}$ the *k-stretch* of p is the path kp given by

$$(kp)(t) = p(kt), \qquad 0 \le t \le 1.$$

Let C be the dominant chamber, let ρ be the half-sum of the positive roots and set

$$C - \rho = \{x - \rho \in \mathfrak{h}_{\mathbb{R}}^* \mid x \in C\}. \qquad \textit{PICTURE}$$

Write $p \subseteq C - \rho$ if $p(t) \in C - \rho$ for all $0 \le t \le 1$. The definitions imply that

$$p \in \vec{B} \text{ is a highest weight path} \qquad \text{if and only if} \qquad p \subseteq C - \rho.$$

Theorem 1.15. *Let B be a subcrystal of $\vec{B}$ such that B_μ is finite for all $\mu \in P$. Then*

$$\chi^B = \sum_{\substack{b \in B \\ b \subseteq C - \rho}} s_{\mathrm{wt}(b)},$$

where s_λ denotes the Weyl character corresponding to $\lambda \in P^+$.

Proof. Let $\mu \in P^+$. Then

$$\chi^B a_\rho\Big|_{a_{\mu+\rho}} = \left(\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho}\right)\left(\sum_{p \in B} e^{\mathrm{wt}(p)}\right)\Big|_{a_{\mu+\rho}} = \sum_{\substack{w \in W \\ p \in B}} (-1)^{\ell(w)} e^{\mathrm{wt}(p) + w\rho}\Big|_{e^{\mu+\rho}}. \tag{1.1}$$

Let $p \in B$ and $w \in W$ be such that $\mathrm{wt}(p) + w\rho = \mu + \rho$. Let t_0 be maximal such that there is an $i \in I$ with $w\rho + p(t_0) \in H_{\alpha_i}$. If t_0 does not exist then $p \in C - \rho$ and $w = 1$. If t_0 does exist set

$$\Phi(p) = \begin{cases} \tilde{f}_i^{-\langle w\rho, \alpha_i^\vee \rangle} p, & \text{if } \langle w\rho, \alpha_i^\vee \rangle < 0, \\ \tilde{e}_i^{\langle w\rho, \alpha_i^\vee \rangle} p, & \text{if } \langle w\rho, \alpha_i^\vee \rangle > 0. \end{cases}$$

Then $\mathrm{wt}(p) + w\rho = \mathrm{wt}(\Phi(p)) + s_i w\rho$ and and the pairs (p, w) and $(\Phi(p), s_i w)$ cancel in the sum (???). □

Let $\lambda \in P^+$ and let b_λ^+ be a highest weight path with $\mathrm{wt}(b_\lambda^+) = \lambda$. For example, b_λ^+ might be the path given by

$$b_\lambda^+(t) = t\lambda, \qquad 0 \le t \le 1.$$

Define

$$B(\lambda) = \{\tilde{f}_{i_k} \cdots \tilde{f}_{i_2} \tilde{f}_{i_1} b_\lambda^+ \mid 1 \le i_1, \ldots, i_k \le n, k \in \mathbb{Z}_{\ge 0}\}$$

so that $b(\lambda)$ is the collection of paths obtained by applying finite sequences of $\tilde{f}_i$ to b_λ^+.

Corollary 1.16. *Let $\lambda \in P^+$.*

(a) $s_\lambda = \sum_{b\in B(\lambda)} e^{\mathrm{wt}(b)}$.

(b) $s_\lambda(q^\rho) = \prod_{\alpha\in R^+} \frac{[\langle\lambda+\rho,\alpha^\vee\rangle]}{[\langle\rho,\alpha^\vee\rangle]} = \sum_{b\in B(\lambda)} q^{\langle \mathrm{wt}(b),\rho\rangle}$

(c) $s_\lambda(1) = \prod_{\alpha\in R^+} \frac{\langle\lambda+\rho,\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle} = \mathrm{Card}(B(\lambda))$.

(d) If $K_{\lambda\mu}$ are defined by $s_\lambda = \sum_{\mu\in P} K_{\lambda\mu} m_\mu$ then $K_{\lambda\mu} = \mathrm{Card}(B(\lambda)_\mu)$,

Proof. □

Let $J \subseteq I$. The crystal

$$\mathrm{Res}_J B \quad \text{is} \quad B \qquad \text{with only those crystal operators} \qquad \tilde e_j, \tilde f_j \quad \text{for } j \in J.$$

This is a crystal for the parabolic subsystem (W_J, C_J, P).

Proposition 1.17. *Let $\lambda \in P^+$ and let $B(\lambda)$ be the irreducible highest weight crystal of highest weight λ.*

$$\mathrm{Res}_J B(\lambda) = \bigoplus_{\substack{p\in B(\lambda)\\ p\subseteq C_J-\rho_J}} B(wt(p)) \qquad B(\lambda)\otimes B(\mu) = \bigoplus_{\substack{p\in B(\mu)\\ p_\lambda\otimes p\subseteq C-\rho}} B(\mathrm{wt}(p)) \qquad B(\lambda)^* \cong B(-w_0\lambda).$$

Proof. □

Corollary 1.18. *Let $\mu,\nu,\lambda \in P^+$ and let $\tau \in P_J^+$.*

$$c^\lambda_{\mu\nu} = \mathrm{Card}(\{b\in B(\mu) \mid \mathrm{wt}(b_\mu^+\otimes b) = \lambda \ \ \textit{and} \ \ b_\mu^+\otimes b \in C-\rho\}), \qquad \textit{and}$$
$$c^\lambda_{J,\tau} = \mathrm{Card}(\{b\in B(\lambda) \mid \mathrm{wt}_J(b) = \tau \ \ \textit{and} \ \ b\subseteq C_J-\rho_J\}).$$

Then

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \qquad \textit{and} \qquad s_\lambda = \sum_{\tau\in P_J^+} c^\lambda_{J,\tau} s^J_\tau.$$

Proof. □

For paths $p_1, p_2 \in \vec B$ define

$$d(p_1,p_2) = \max\{|p_1(t)-p_2(t)| \mid 0\le t\le 1\}.$$

Proposition 1.19. *The operators*

$$\tilde e_i \colon \vec B \to \vec B \cup \{0\} \qquad \textit{and} \qquad \tilde f_i \colon \vec B \to \vec B\cup\{0\}$$

are the unique operators such that

(1)

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \qquad \text{and}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

(2) The operators

$$s_i p = \begin{cases} \tilde{f}_i^{\langle \mathrm{wt}(p), \alpha_i^\vee \rangle} p, & \text{if } \langle \mathrm{wt}(p), \alpha_i^\vee \rangle > 0, \\ p, & \text{if } \langle \mathrm{wt}(p), \alpha_i^\vee \rangle = 0, \\ \tilde{e}_i^{\langle \mathrm{wt}(p), \alpha_i^\vee \rangle} p, & \text{if } \langle \mathrm{wt}(p), \alpha_i^\vee \rangle < 0, \end{cases}$$

define an action of W *on* $\vec{B}$.

(3) $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq 0\}$ *and* $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq 0\}$.

(4) If $\tilde{e}_i b \neq 0$ *then* $\tilde{f}_i \tilde{e}_i b = b$ *and* $(\tilde{e}_i b)^* = \tilde{f}_i b^*$.
If $\tilde{f}_i b \neq 0$ *then* $\tilde{e}_i \tilde{f}_i b = b$ *and* $(\tilde{f}_i b)^* = \tilde{e}_i b^*$.

(5) If $b \in \vec{B}$ *and* $k \in \mathbb{Z}_{>0}$ *then*

$$k(\tilde{e}_i b) = \tilde{e}_i^k (kb) \qquad \text{and} \qquad k(\tilde{f}_i b) = \tilde{f}_i^k (kb).$$

(6) There is a constant c*, depending only on* (W, C, P) *such that*
If $\tilde{e}_i b_1 \neq 0$ *and* $\tilde{e}_i b_2 \neq 0$ *then* $d(\tilde{e}_i b_1, \tilde{e}_i b_2) < c \cdot d(b_1, b_2)$, *and*
If $\tilde{f}_i b_1 \neq 0$ *and* $\tilde{f}_i b_2 \neq 0$ *then* $d(\tilde{f}_i b_1, \tilde{f}_i b_2) < c \cdot d(b_1, b_2)$.

Proof. □

1.6 The crystal $B_n^{\otimes k}$

Let

$$\begin{aligned} \mathbb{R}^n &= \sum_{i=1}^n \mathbb{R}\varepsilon_i, \qquad \text{with the } \varepsilon_i \text{ an orthonormal basis,} \\ W &= S_n, \quad \text{acting on } \mathbb{R}^n \text{ by permuting the } \varepsilon_i, \\ C &= \{\mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n \mid \mu_1 < \mu_2 < \cdots < \mu_n\}, \\ L &= \sum_{i=1}^n \mathbb{Z}\varepsilon_i, \end{aligned}$$

and let b_i be the straightline path from 0 to ε_i. Then

$$B_n^{\otimes k} = \{b_{i_1} \cdots b_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$$

is the set of length k paths in $\mathbb{R}^n$ where each step is a unit step in one of the directions $\varepsilon_1, \ldots, \varepsilon_n$. To compute the operators $\tilde{e}_i$ and $\tilde{f}_i$ on a path $b = b_{i_1} \cdots b_{i_k}$ in $B_n^{\otimes k}$ place the value

$$\begin{aligned} +1 &= \langle \varepsilon_i, \alpha_i^\vee \rangle \text{over each } b_i, \\ -1 &= \langle \varepsilon_{i+1}, \alpha_i^\vee \rangle \text{over each } b_{i+1}, \\ 0 &= \langle \varepsilon_j, \alpha_i^\vee \rangle \text{over each } b_j,\ j \neq i, i+1, \end{aligned}$$

Ignoring 0s read this sequence of ± 1 from left to right and succesively remove adjacent $(+1,-1)$ pairs until the sequence is of the form

$$\underbrace{+1\ +1\ \ldots +\overset{\substack{\text{cogood}\\ \downarrow}}{1}}_{\text{conormal nodes}}\quad \underbrace{\overset{\substack{\text{good}\\ \downarrow}}{-1}\ -1\ \ldots -1}_{\text{normal nodes}}$$

The -1s in this sequence are the *normal nodes* and the $+1$s are the *conormal nodes*. The *good node* is the leftmost normal node and the *cogood node* is the right most conormal node. The good node is exactly at the position where the path b is at its most negative point with respect to the hyperplane H_{α_i}. Then

$$\begin{aligned}
\varepsilon_i(b) &= (\#\text{ of normal nodes}),\\
\varphi_i(b) &= (\#\text{ of conormal nodes}),\\
\tilde{e}_i(b) &= \text{same as } b \text{ except with the cogood node path step changed to } b_i,\\
\tilde{f}_i(b) &= \text{same as } b \text{ except with the good node path step changed to } b_i,
\end{aligned}$$

For example, if $n=5$ and $k=30$ and

$$b = b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3b_2b_1b_1b_2b_3b_3b_2b_1b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4$$

then the parentheses in the table

$$\begin{array}{cccccccccc}
 & & & (&) & -1 & & & (&) \\
 & & -1 & (& (&) & & &) & +1 \\
 & & & +1 & +1 & (& (&) &) & 0 \\
0 & 0 & 0 & +1 & -1 & -1 & 0 & 0 & +1 & -1 \\
0 & 0 & -1 & +1 & +1 & -1 & 0 & 0 & -1 & +1 \\
0 & 0 & 0 & +1 & +1 & +1 & +1 & -1 & -1 & 0 \\
b_4 & b_3 & b_3 & b_1 & b_2 & b_2 & b_4 & b_4 & b_1 & b_2 \\
b_3 & b_3 & b_2 & b_1 & b_1 & b_2 & b_3 & b_3 & b_2 & b_1 \\
b_4 & b_5 & b_5 & b_1 & b_1 & b_1 & b_1 & b_2 & b_2 & b_4
\end{array}$$

indicate the $(+1,-1)$ pairings and the numbers in the top row indicate the resulting sequence of -1s and $+1$s. Then

$$\begin{aligned}
&\varepsilon_1(b) = 2, \qquad \varphi_1(b) = 3,\\
&\tilde{e}_i(b) = b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3b_2b_1b_1b_2b_3b_3b_2\underline{b_2}b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4,\\
&\tilde{f}_i(b) = b_4b_3b_3b_1b_2b_2b_4b_4b_1b_2b_3b_3\underline{b_1}b_1b_1b_2b_3b_3b_2b_1b_4b_5b_5b_1b_1b_1b_1b_2b_2b_4,
\end{aligned}$$

The highest weight paths in $B_n^{\otimes k}$ are the $b = b_{i_1}\cdots b_{i_k}$ such that for every $1 \le j \le k$ and every $1 \le i \le n$ the

$$(\#\text{ of } b_i \text{ in } b_{i_1}\cdots b_{i_j}) \ \ge\ (\#\text{ of } b_{i+1} \text{ in } b_{i_1}\cdots b_j).$$

The map

$$Q\colon \{\text{highest weight paths in } B_n^{\otimes k}\} \xleftrightarrow{1-1} \{\text{standard tableaux with } k \text{ boxes and} \le n \text{ rows}\}$$

is given by making the standard tableau

$$Q(b) \qquad \text{such that} \qquad \text{entry } j \text{ is in row } i \text{ if } b \text{ has } b_i \text{ in position } j \text{ (i.e. if } b_{i_j} = b_i).$$

For example, if $k = n = 4$ then $B_4 = \{b_1, b_2, b_3, b_4\}$ and the map Q is given by

$$b_1b_1b_1b_1 \longmapsto \begin{array}{l}1234\end{array} \qquad\qquad b_1b_2b_3b_4 \longmapsto \begin{array}{l}1\\2\\3\\4\end{array}$$

$$b_1b_1b_1b_2 \longmapsto \begin{array}{l}123\\4\end{array} \qquad\qquad b_1b_1b_2b_3 \longmapsto \begin{array}{l}12\\3\\4\end{array}$$

$$b_1b_1b_2b_1 \longmapsto \begin{array}{l}124\\3\end{array} \qquad\qquad b_1b_2b_1b_3 \longmapsto \begin{array}{l}13\\2\\4\end{array}$$

$$b_1b_2b_1b_1 \longmapsto \begin{array}{l}134\\2\end{array} \qquad\qquad b_1b_2b_3b_1 \longmapsto \begin{array}{l}14\\2\\3\end{array}$$

$$b_1b_1b_2b_2 \longmapsto \begin{array}{l}12\\34\end{array} \qquad\qquad b_1b_2b_1b_2 \longmapsto \begin{array}{l}13\\24\end{array}$$

Let $x_i = e^{\varepsilon_i}$. For a highest weight path $b \in B_n^{\otimes k}$

$$\mathrm{wt}(b) = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \qquad \text{if the shape of } Q(b) \text{ is} \qquad \lambda = (\lambda_1, \ldots, \lambda_n).$$

and so the character of the crystal $B_n^{\otimes k}$ is

$$(x_1 + \cdots + x_n)^k = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le n}} f^\lambda s_\lambda, \qquad \text{where} \qquad f^\lambda = (\#\text{ of standard tableaux of shape } \lambda).$$

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