The fundamental data and Weyl groups

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1.1 The fundamental data

The fundamental data is a fundamental region which is invariant under dilations. The cases we study in this book are when the group is acting linearly on some $\mathbb{K}^n$ where $\mathbb{K}$ is a commutative ring of characteristic 0. Invariant under dilations means that if the fundamental region is multiplied by a nonzero nonunit in $\mathbb{K}$ it is still isomorphic to itself, i.e. $(rA \subseteq (r\mathbb{K})^n) \cong (A \subseteq \mathbb{K}^n)$

If $\mathbb{K} = \mathbb{Z}$ then the fundamental data is an affine Weyl group and the alcove is the dual graph of the extended Dynkin diagram. In this case the fundamental is equivalent to the data (W, C, P) where

W is a finite real reflection group,

C is a fixed fundamental chamber, and

P is a W-invariant lattice.

By a theorem of Chevalley, the data (W, C, P) is equivalent to the data

G, a complex reductive algebraic group,

B, a Borel subgroup of G,

T, a maximal torus of G contained in B.

1.2 Weyl groups

A *lattice* is a free $\mathbb{Z}$-module. Let P be a lattice with a ($\mathbb{Z}$-linear) action of a finite group W. Thus P is a module for the group algebra $\mathbb{Z}W$. Extending coefficients, define

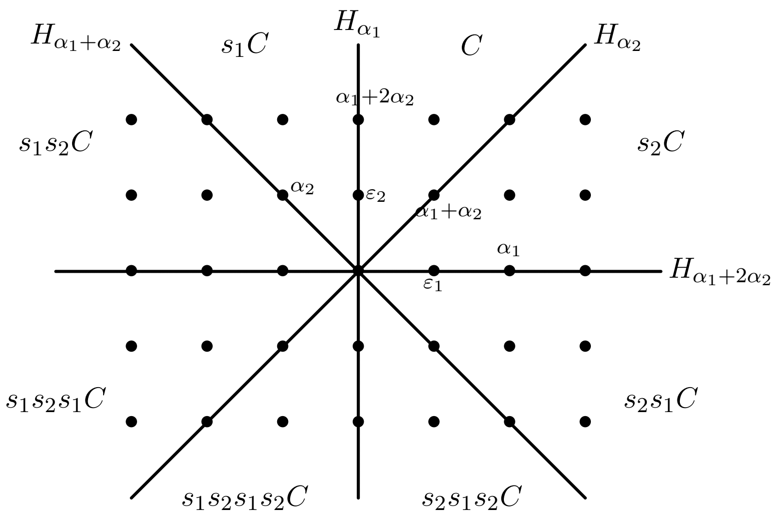
$$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} P \qquad \text{and} \qquad \mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}},$$

so that $\mathfrak{h}^*_{\mathbb{R}}$ and $\mathfrak{h}^*$ are vector spaces which are modules for the group algebras $\mathbb{R}W$ and $\mathbb{C}W$, respectively. Assume that $\mathfrak{h}^*_{\mathbb{R}}$ is an irreducible W-module, that the action of W on $\mathfrak{h}^*_{\mathbb{R}}$ has fundamental regions, and

$$\text{fix a fundamental region } C \text{ for the action of } W \text{ on } \mathfrak{h}^*_{\mathbb{R}}. \tag{1.1}$$

An example is when the lattice $P = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2$ with $\{\varepsilon_1, \varepsilon_2\}$ an orthonormal basis of $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^2$ and $W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$ is the dihedral group of order 8 generated by the reflections s_1 and s_2 in the hyperplanes H_{α_1} and H_{α_2}, respectively, where

$$H_{\alpha_1} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \varepsilon_1\rangle = 0\}, \qquad \text{and} \qquad H_{\alpha_2} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \varepsilon_2 - \varepsilon_1\rangle = 0\}.$$



This is *type* C_2.

Define

$$P^+ = P \cap \overline{C} \qquad \text{and} \qquad P^{++} = P \cap C$$

so that P^+ is a set of representatives of the orbits of the action of W on P. The *fundamental weights* are the generators $\omega_1, \ldots, \omega_n$ of the $\mathbb{Z}_{\geq 0}$-module P^+ so that

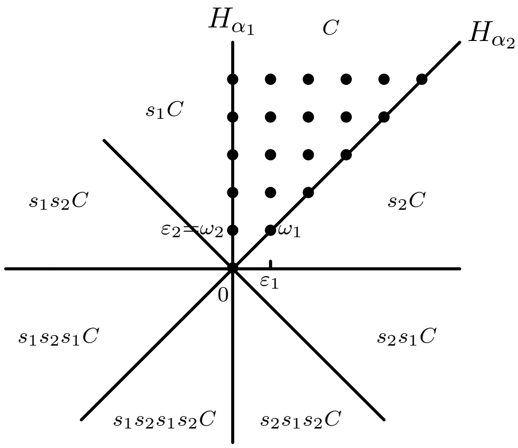
$$C = \sum_{i=1}^n \mathbb{R}_{\geq 0}\omega_i, \qquad P^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i, \qquad \text{and} \qquad P^{++} = \sum_{i=1}^n \mathbb{Z}_{>0}\omega_i. \tag{1.2}$$

The lattice P has $\mathbb{Z}$-basis $\omega_1, \ldots, \omega_n$ and the map

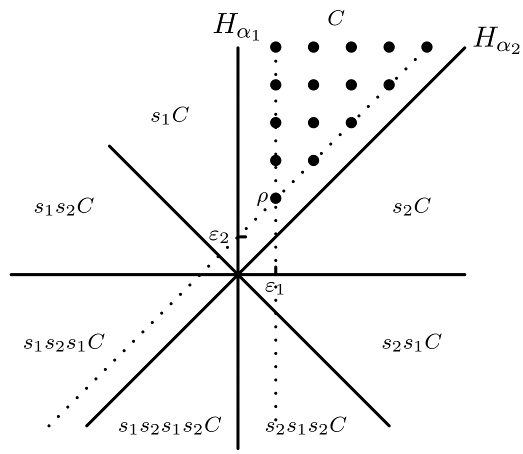
$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \rho + \lambda \end{array} \qquad \text{where} \quad \rho = \omega_1 + \ldots + \omega_n. \tag{1.3}$$

is a bijection.

In the case of type C_2 the picture is



The set P^+



The set P^{++}

with

$$\begin{aligned} \omega_1 &= \varepsilon_1+\varepsilon_2, & \alpha_1 &= 2\varepsilon_1, & \alpha_1^\vee &= \varepsilon_1, \\ \omega_2 &= \varepsilon_2, & \alpha_2 &= \varepsilon_2-\varepsilon_1, & \alpha_2^\vee &= \alpha_2, \end{aligned}$$

and

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1+2\alpha_2)\}.$$

Let $\langle,\rangle\colon \mathfrak{h}_\mathbb{R}^*\times\mathfrak{h}_\mathbb{R}^* \to \mathbb{R}$ be a nondegenerate W-invariant symmetric bilinear form on $\mathfrak{h}_\mathbb{R}^*$. Any symmetric bilinear form $(,)\colon \mathfrak{h}_\mathbb{R}^*\times\mathfrak{h}_\mathbb{R}^*\to\mathbb{R}$ can be made into a W-invariant form $\langle,\rangle$ by defining

$$\langle x,y\rangle = \sum_{w\in W}(wx,wy), \qquad \text{for } x,y,\in\mathfrak{h}_\mathbb{R}^*.$$

The *simple coroots* are $\alpha_1^\vee,\ldots,\alpha_n^\vee$ the dual basis to the fundamental weights,

$$\langle\omega_i,\alpha_j^\vee\rangle = \delta_{ij}. \tag{1.4}$$

Define

$$\overline{C^\vee} = \sum_{i=1}^n \mathbb{R}_{\le 0}\alpha_i^\vee \qquad \text{and} \qquad C^\vee = \sum_{i=1}^n \mathbb{R}_{<0}\alpha_i^\vee. \tag{1.5}$$

The *dominance order* is the partial order on $\mathfrak{h}_\mathbb{R}^*$ given by

$$\mu\le\lambda \qquad \text{if} \qquad \mu\in\lambda+\overline{C^\vee}. \tag{1.6}$$

PICTURE

References

An alternate reference for the material in this section is [Bou] (Bourbaki Chpt. IV-VI).