Murphy elements for Temperley-Lieb algebras

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1 The Temperley-Lieb algebra

The *Temperley-Lieb algebra* is the subalgebra of the partition algebra with basis $\{d \in A_k \mid d \text{ is planar}\}$.

The Temperley-Lieb algebra $\mathbb{C}T_k(n)$ is presented by generators $e_1, e_2, ..., e_{k-1}$ and relations

$$\begin{array}{ll} e_i e_j = e_j e_i, & \text{if } |i-j| > 1, \\ e_i e_{i\pm 1} e_i = e_i, & \text{and} \\ e_i^2 = n e_i. & \end{array}$$

If

$$[2] = q + q^{-1} = n \qquad \text{then} \qquad q = \frac{1}{2}(n + \sqrt{n^2-4}), \quad q^{-1} = \frac{1}{2}(n - \sqrt{(n^2-4)}),$$

since $q^2 - nq + 1 = 0$. Then

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{1}{2^{k-1}} \sum_{m=1}^{(k+1)/2} \binom{k}{2m-1} n^{k-2m+1} (n^2-4)^{m-1}.$$

The problem with this expression is that it is not clear that $[k]$ is a polynomial in n with integer coefficients (which alternate in sign?).

The *affine Hecke algebra* $\tilde{H}_k$ is given by generators

$$X^{\varepsilon_1} \qquad \text{and} \qquad T_1, \ldots, T_{n-1},$$

with relations

$$\begin{array}{ll} X^{\varepsilon_1} T_1 X^{\varepsilon_1} T_1 = T_1 X^{\varepsilon_1} T_1 X^{\varepsilon_1} & \\ X^{\varepsilon_1} T_i = T_i X^{\varepsilon_1}, & \text{for } i > 1, \\ T_i T_j = T_j T_i, & \text{if } |i-j| > 1, \\ T_i T_{i\pm 1} T_i = T_{i+1} T_i T_{i+1}, & \text{if } 2 \le i \le k-1, \\ T_i^2 = (q - q^{-1}) T_i + 1. & \end{array} \tag{1.1}$$

If

$$X^{\varepsilon_i} = T_{i-1} \cdots t_2 T_1 X^{\varepsilon_1} T_1 T_2 \cdots T_{i-1} \qquad \text{then} \qquad X^{\varepsilon_i} X^{\varepsilon_j} = X^{\varepsilon_j} X^{\varepsilon_i}, \quad \text{for } 1 \le i, j \le k.$$

The affine Hecke algebra acts on $V^{\otimes k}$ by

$$X^{\varepsilon_1} = \mathrm{id} \qquad \text{and} \qquad T_j = \mathrm{id}^{\otimes (j-1)} \otimes T \otimes \mathrm{id}^{\otimes (k-(j+1))}$$

where

$$T(v_{i_1} \otimes v_{i_2}) = \begin{cases} v_{i_2} \otimes v_{i_1}, & \text{if } i_1 \le i_2, \\ v_{i_2} \otimes v_{i_1} + (q - q^{-1}) v_{i_1} \otimes v_{i_2}, & \text{if } i_1 > i_2. \end{cases}$$

If we set $e_i = q - T_i$ and $\dim(V) = 2$ then $\mathbb{C}T_k(n)$ acts on $V^{\otimes k}$ by

$$e(v_{i_1} \otimes v_{i_2}) = \begin{cases} q v_{i_1} \otimes v_{i_2} - v_{i_2} \otimes v_{i_1}, & \text{if } i_1 \le i_2, \\ q^{-1} v_{i_1} \otimes v_{i_2} - v_{i_2} \otimes v_{i_1}, & \text{if } i_1 > i_2. \end{cases}$$

The *Iwahori-Hecke algebra* is the subalgebra of $\tilde{H}_k$ generated by $T_1, \ldots, T_k$. Letting $n = q + q^{-1}$, There are surjective algebra homomorphism

$$\begin{array}{ccccc} \tilde{H}_k(q) & \longrightarrow & H_k(q) & \longrightarrow & T_k(n) \\ X^{\varepsilon_1} & \longmapsto & 1 & & \\ T_i & \longmapsto & T_i & \longmapsto & e_i - q^{-1} \end{array}$$

If

$$T_i = e_i - q^{-1}$$

then a direct calculation shows that $e_i^2 = (q + q^{-1}) e_i$

$$e_1 e_2 e_1 = e_1 \qquad \text{if and only if} \qquad q^{-3} + q^{-2} T_1 + q^{-2} T_2 + q^{-1} T_1 T_2 + q^{-1} T_2 T_1 + T_1 T_2 T_1 = 0$$

and similarly

$$e_2 e_1 e_2 = e_2 \qquad \text{if and only if} \qquad q^{-3} + q^{-2} T_1 + q^{-2} T_2 + q^{-1} T_1 T_2 + q^{-1} T_2 T_1 + T_2 T_1 T_2 = 0$$

There is an alternative surjective homormphism

$$\begin{array}{ccccc} \tilde{H}_k(q) & \longrightarrow & H_k(q) & \longrightarrow & T_k(n) \\ X^{\varepsilon_1} & \longmapsto & 1 & & \\ T_i & \longmapsto & T_i & \longmapsto & q - e_i \end{array}$$

with $\ker \psi$ generated by

$$q^{-3} - q^{-2} T_1 - q^{-2} T_2 + q^{-1} T_1 T_2 + q^{-1} T_2 T_1 - T_2 T_1 T_2 = 0$$

Let us write

$$T_i = e_i - q^{-1}, \qquad \text{and let} \quad X^{\varepsilon_1} = 1, \quad \text{and} \quad X^{\varepsilon_i} = T_{i-1} X^{\varepsilon_{i-1}} T_{i-1}$$

in the Temperley-Lieb algebra. Then define $m_1, \ldots, m_k$ by

$$m_1 = 0 \qquad \text{and} \qquad (q - q^{-1}) m_i = q^{i-2} X^{\varepsilon_i} - q^{i-4} X^{\varepsilon_{i-1}} \quad \text{for } 2 \le i \le k. \tag{1.2}$$

Solving for X^{ε_i} in terms of the m_i gives

$$X^{\varepsilon_i} = (q - q^{-1})(q^{-(i-2)} m_i + q^{-(i-2+1)} m_{i-1} + \cdots + q^{-(2i-4)} m_2) + q^{-2(i-1)}, \tag{1.3}$$

from which one obtains

$$q^{(k-2)}(X^{\varepsilon_1} + X^{\varepsilon_2} + \cdots + X^{\varepsilon_k}) - q[k] = (q - q^{-1})(m_k + [2] m_{k-1} + \cdots + [k-1] m_2). \tag{1.4}$$

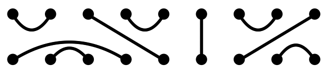
Substituting for $X^{\varepsilon_{i-1}}$ in terms of the m_i in

$$(q-q^{-1})m_i = q^{i-2}X^{\varepsilon_i} - q^{i-4}X^{\varepsilon_{i-1}} = q^{i-2}(E_{i-1}-q^{-1})X^{\varepsilon_{i-1}}(E_{i-1}-q^{-1}) - q^{i-4}X^{\varepsilon_{i-1}}$$

gives

$$\begin{aligned} m_i &= q^{-(i-2)}E_{i-1} + qE_{i-1}m_{i-1}E_{i-1} - (E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\ &\quad + (q-q^{-1})(m_{i-2} + q^{-1}m_{i-3} + q^{-2}m_{i-4} + \cdots + q^{-(i-4)}m_2)E_{i-1}. \end{aligned} \tag{1.5}$$

The *cycle type* of a diagram $d \in T_k$ is the set partition $\tau(d)$ of $\{1,2,\ldots,k\}$ obtained from d by setting $1 = 1', 2 = 2', \ldots, k = k'$. Since d is planar this is a set partition of the form $\{\{1,2,\ldots,\mu_1\},\{\mu_1+1,\mu_1+2,\ldots,\mu_1+\mu+2\},\ldots,\{\mu_1+\cdots+\mu_{\ell-1}+1,\ldots,k\}\}$ where $(\mu_1,\ldots,\mu_\ell)$ is a composition of k and we will simplifying notation by writiing $\tau(d) = (\mu_1,\ldots,\mu_\ell)$. For example

$d =$  has $\quad \tau(d) = (5,1,3)$.

If $\mu = (\mu_1,\ldots,\mu_\ell)$ is a composition of k define

$$d_\mu = \sum_{\substack{d\in T_k \\ \tau(d)=\mu}} d. \tag{1.6}$$

Theorem 1.1. *With notation as in ???, define elements $m_1,\ldots,m_k \in \mathbb{C}T_k$ by $m_1 = 0$ and*

$$m_i = \sum_{a=0}^{i-2}(-1)^{i-a}[a+1]d_{1^a\ell} + \sum_{j,\ell>1}(-1)^{i-a-b-1}[a+1]([b+2]-[b])d_{1^aj1^b\ell}, \qquad \text{for } i>1.$$

Then

(a) $m_im_j = m_jm_i$ *for all* $1 \le i,j \le n$.

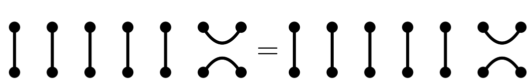
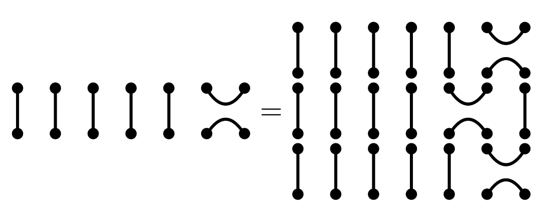
(b) $m_k + [2]m_{k-1} + \cdots + [k-1]m_2$ *is a central element of* $\mathbb{C}T_k(n)$ *which acts on the irreducible representation labeled by* $\lambda = (2^b1^{k-2b})$ *by the constant*

$$-[k], \quad \text{if } b=0, \quad 0, \quad \text{if } b=1, \quad \sum_{i=1}^{b-1}[k-2i], \quad \text{if } b>1.$$

(c)

$$\begin{aligned} m_i &= q^{-(i-2)}E_{i-1} + qE_{i-1}m_{i-1}E_{i-1} - (E_{i-1}m_{i-1} + m_{i-1}E_{i-1}) \\ &\quad + (q-q^{-1})(m_{i-2} + q^{-1}m_{i-3} + q^{-2}m_{i-4} + \cdots + q^{-(i-4)}m_2)E_{i-1}. \end{aligned}$$

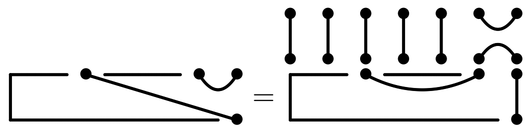
Proof. (a) The proof is by induction on n using the formula (???). *Case 1:* The coefficient of a term d_{1^a2} in m_i gets a contribution from the term $q^{-(i-2)}E_{i-1}$ and a contribution from one term of $qE_{i-1}d_{1^{i-3}2}E_{i-1}$ in $qE_{i-1}m_{i-1}E_{i-1}$

 and 

resulting in the value

$$q^{-(i-2)} + q(-1)^{i-1-(i-3)}[i-2] = q^{-(i-2)} + q[i-2] = [i-1].$$

Case 2: Let $\ell > 2$. The coefficient of a term in $d_{1^a\ell}$ in m_i gets a contribution only from one term of $-E_{n-1}d_{1^a(\ell-1)}$ in $-E_{i-1}m_{i-1}$ or one term of $-d_{1^a(\ell-1)}E_{i-1}$ in $-m_{i-1}E_{i-1}$



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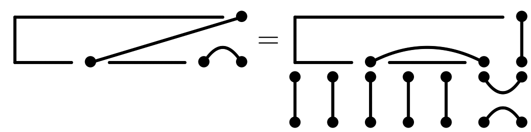
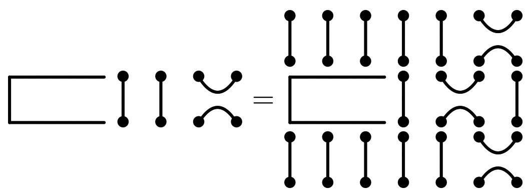


resulting in the value

$$-(-1)^{i-1-a}[a+1] = (-1)^{i-a}[a+1].$$

Case 3a: Let $b > 0$ and $j > 1$. The coefficient of a term in $d_{1^aj1^b2}$ in m_i gets a contribution from one term of $qE_{i-1}d_{1^aj1^{b-1}2}E_{i-1}$ in $qE_{i-1}m_{i-1}E_{i-1}$ and a contribution from one term of $(q-q^{-1})q^{-b}d_{1^aj}E_{i-1}$ in $(q-q^{-1})q^{-b}m_{i-2-b}E_{i-1}$



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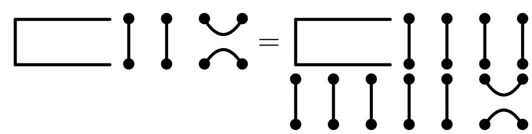
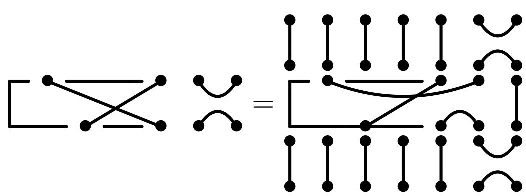


resulting in the value

$$\begin{aligned} &q(-1)^{i-1-a-(b-1)-1}[a+1]([b+1]-[b-1]) + q^{-b}(q-q^{-1})(-1)^{i-2-b-a}[a+1] \\ &\qquad = (-1)^{i-2-a-b}[a+1](-1)([b+2]-[b]). \end{aligned}$$

Case 3b: Let $b = 0$ and $j > 1$. The coefficient of a term in $d_{1^aj1^b2} = d_{1^aj2}$ in m_i gets a contribution from two terms of $qE_{i-1}d_{1^a(j+1)}E_{i-1}$ in $qE_{i-1}m_{i-1}E_{i-1}$



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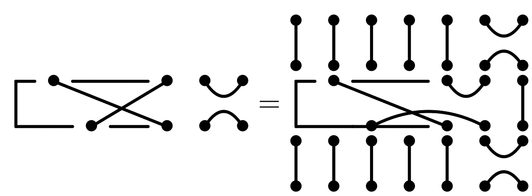
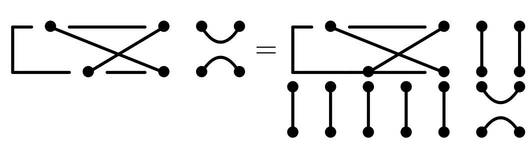


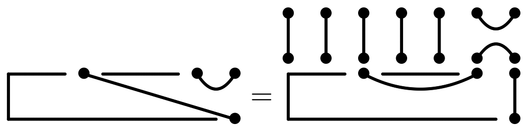
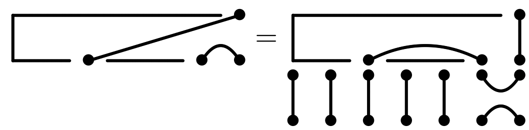
and a contribution from one term of $(q-q^{-1})q^{-b}d_{1^aj}E_{i-1}$ in $(q-q^{-1})q^{-b}m_{i-2}E_{i-1}$



resulting in the value

$$\begin{aligned} &2q(-1)^{i-1-a}[a+1] + (q-q^{-1})(-1)^{i-2-a}[a+1] \\ &\qquad = (q+q^{-1})(-1)^{i-2-a}[a+1] = (-1)^{i-2-a-0}([0+2]-[0]). \end{aligned}$$

Case 4: Let $b \geq 0$ and $j > 1$. The coefficient of a term in $d_{1^a j 1^b \ell}$ in m_i gets a contribution only from one term of $-E_{i-1}d_{1^a j 1^b(\ell-1)}$ in $-E_{i-1}m_{i-1}$ or from one term of $-d_{1^a j 1^b(\ell-1)}E_{i-1}$ in $-m_{i-1}E_{i-1}$

 or 

resulting in the value

$$-(-1)^{i-1-a-b-1}[a+1]([b+2]-[b]) = (-1)^{i-a-b-1}[a+1]([b+2]-[b]).$$

(b) This result follows from (???) and the calculation

$$\begin{aligned} q^{k-2}\sum_{b\in\lambda} q^{2\binom{b}{}} - q[k] &= q^{k-2}\big(q^0+q^{-2}+\cdots+q^{-2(k-b-1)}\big) + \big(q^2+q^0+\cdots+q^{-2(b-2)}\big) \\ &\qquad - q(q^{k-1}+q^{k-3}+\cdots+q^{-(k-1)}) \\ &= \sum_{i=1}^{b-1}(q^{k-2i}-q^{-(k-2i)}). \end{aligned}$$

$\square$

For n such that $\mathbb{C}T_k(n)$ is semisimple, the simple $T_k(n)$ are indexed by partitions in the set

$$\hat{T}_k = \{\lambda \vdash k \mid \lambda \text{ has at most two columns}\}.$$

The irreducible $\mathbb{C}T_k(n)$ modules have seminormal basis

$$\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$$

and

$$X^{\varepsilon_i}v_T = q^{2c(T(i))}v_T.$$

Since $c(T(i)) = c(T(i-1)) - 1$ if the boxes $T(i)$ and $T(i-1)$ are in the same column and $c(T(i)) + c(T(i-1)) = 3 - i$ if the boxes $T(i)$ and $T(i-1)$ are in different columns it follows that

$$m_i v_T = \frac{q^{i-2}q^{2c(T(i))} - q^{i-4)}q^{2c(T(i-1))}}{q-q^{-1}} = c_T(i)v_T,$$

where

$$c_T(i) = \begin{cases} 0, & \text{if } T(i) \text{ and } T(i-1) \text{ are in the same column,} \\ [i-2+2c(T(i))], & \text{if } T(i) \text{ and } T(i-1) \text{ are in different columns.} \end{cases}$$

Build a graph $\hat{T}$ by setting

$$\begin{aligned} &\text{vertices on level } k: \qquad \hat{S}_k = \{\text{rows } \lambda \text{ of lengths } k, k-2, k-4, \ldots\}, \quad \text{and} \\ &\text{an edge } \lambda \to \mu,\ \lambda \in \hat{T}_k,\ \mu \in \hat{T}_{k+1} \text{ if } \mu \text{ is obtained from } \lambda \text{ by adding or removing a box.} \end{aligned} \tag{1.7}$$

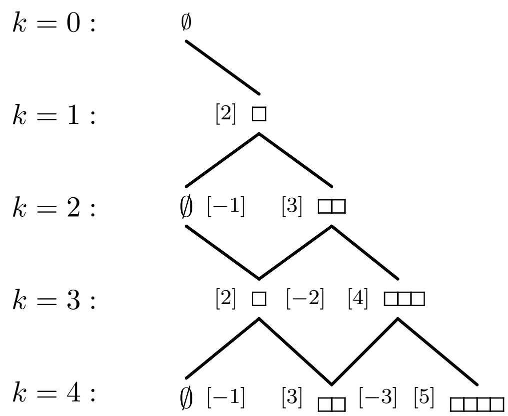
Theorem 1.2. *Define elements* $m_1, \ldots, m_k \in \mathbb{C}T_k$ *by* $m_1 = 0$ *and*

$$m_i = \sum_{a=0}^{i-2} (-1)^{i-a}[a+1]\{1^a \ell\} + \sum_{j,\ell>1} (-1)^{i-a-b-1}[a+1]([b+2]-[b])\{1^a j 1^b \ell\}\,, \qquad \textit{for } i > 1.$$

Then

(a) $m_i m_j = m_j m_i$ *for* $1 \le i, j \le n$.

(b) The eigenvalues of the elements m_i *are given by the diagram*



in the sense that if

$\hat{T}_k$ *is the set of vertices on level* k,
for $\lambda \in \hat{T}_k$, *let* $\hat{T}_k^\lambda = \{$*paths* $p = (\emptyset \to \cdots \to \lambda)$ *to* λ *in* $\hat{T}\}$,

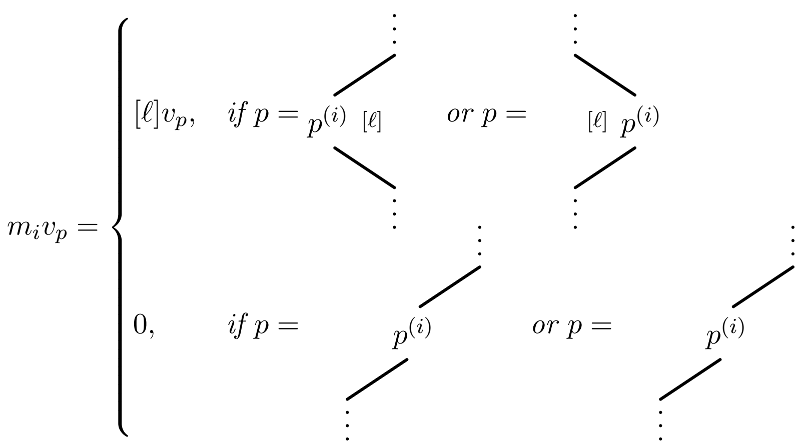
then, for infinitely many values of n,

$\hat{T}_k$ *is an index set for the simple* $\mathbb{C}T_k$ *modules,* T_k^λ,

and

T_k^λ *has a basis* $\{v_p \mid p \in \hat{T}_k^\lambda\}$

with



where $p^{(i)}$ *is the partition on level* i *of the path* p.

(c) $\kappa = m_k + [2]m_{k-1} + \cdots + [k-1]m_2$ *is a central element of* $\mathbb{C}T_k(n)$ *and*

$$\kappa \text{ acts on } T_k^\lambda \text{ by the constant } \quad \begin{cases} -[k], & \text{if } \lambda = \underbrace{\square\square\square\square\square\square\square}_{k \text{ boxes}}, \\ 0, & \text{if } \lambda = \underbrace{\square\square\square\square\square\square\square}_{k-2 \text{ boxes}}, \\ \displaystyle\sum_{i=1}^{b-1}[k-2i], & \text{if } \lambda = \underbrace{\square\square\square\square\square\square\square}_{k-2b \text{ boxes}}. \text{ with } b > 1. \end{cases}$$

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