The symmetric group and Brauer algebras

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1 The symmetric group $\mathbb{C}S_k$ and the Brauer algebra

If $\lambda \in \hat{G}L_n$ then

$$L_{\mathfrak{gl}_n}(\lambda) \otimes V \;\cong\; \bigoplus_{\mu/\lambda=\square} L_{\mathfrak{gl}_n}(\mu) \qquad \text{as } \mathfrak{gl}_n(\mathbb{C})\text{-modules}, \tag{1.1}$$

where the sum is over $\mu \in \hat{\mathfrak{gl}}_n$ that are obtained from λ by adding a box. The *Young lattice* is the graph $\hat{S}$ given by setting

$$\begin{array}{ll} \text{vertices on level } k: & \hat{S}_k = \{\text{partitions } \lambda \text{ with } k \text{ boxes}\}, \quad \text{and} \\ \text{a labeled edge } \lambda \xrightarrow{c(\mu/\lambda)} \mu,\ \lambda \in \hat{S}_k,\ \mu \in \hat{S}_{k+1} & \text{if } \mu \text{ is obtained from } \lambda \text{ by adding a box.} \end{array} \tag{1.2}$$

It encodes the decompositions in (???).

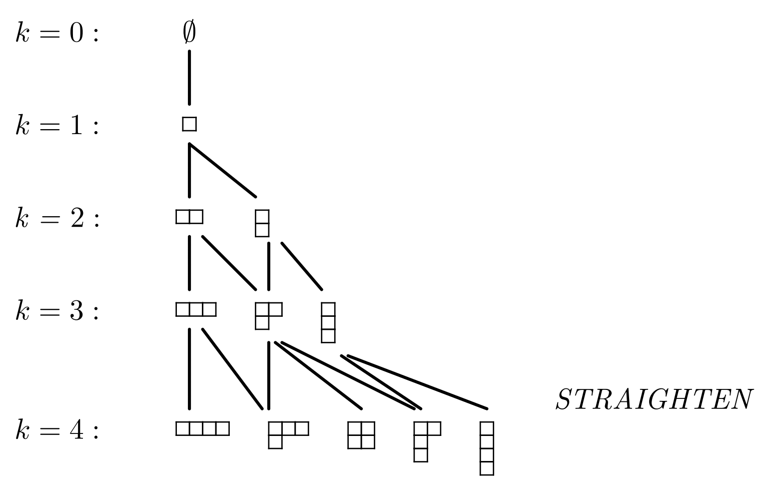
Theorem 1.1. *Define elements $m_1, \ldots, m_k \in \mathbb{C}S_k$ by*

$$m_1 = 0, \qquad \textit{and} \qquad m_i = \sum_{\ell=1}^{i-1} s_{\ell i}, \qquad \textit{for } i > 1.$$

Then

(a) $m_i m_j = m_j m_i$ for $1 \le i, j \le n$.

(b) The eigenvalues of the elements m_i are given by the diagram $\hat{S}$



in the sense that if

$\hat{S}_k$ *is the set of vertices on level* k, *and*

$\hat{S}_k^\lambda = \{$*paths* $p = (\emptyset \to p^{(1)} \to p^{(2)} \to \cdots \to p^{(k)} = \lambda)$ *to* λ *in* $\hat{S}\}$, *for* $\lambda \in \hat{S}_k$,

then

$$\hat{S}_k \text{ is an index set for the simple } \mathbb{C}S_k \text{ modules } S_k^\lambda \qquad \text{and}$$

$$S_k^\lambda \qquad \text{has a basis} \qquad \{v_p \mid p \in \hat{S}_k^\lambda\} \qquad \text{with} \qquad m_i v_p = c(p(i)) v_p,$$

where $p(i) = p^{(i)}/p^{(i-1)}$ *is the box added at step* i *in* p *and* $c(b)$ *denotes the content of the box* b.

(c) $\kappa = m_k + m_{k-1} + \cdots + m_2$ *is a central element of* $\mathbb{C}S_k$ *and*

$$\kappa \text{ acts on } S_k^\lambda \text{ by the constant} \qquad \sum_{b\in\lambda} c(b).$$

Proof. The tensor product rule for GL_n is

$$L_{\mathfrak{gl}_n}(\mu) \otimes V \cong \bigoplus_{\lambda/\mu=\square} L_{\mathfrak{gl}_n}(\mu),$$

where the sum is over all partitions λ such that $\ell(\lambda) \le n$, $\lambda \supseteq \mu$ and λ differs from μ by a single box. Since the S_k action and the GL_n action commute on $V^{\otimes k}$ it follows that,

$$\text{as } (U\mathfrak{gl}_n, \mathbb{C}S_k) \text{ bimodules,} \qquad V^{\otimes k} \cong \bigoplus_{\substack{\lambda\vdash k\\ \ell(\lambda)\le n}} L_{\mathfrak{gl}_n}(\lambda) \otimes S_k^\lambda,$$

where S_k^λ are some S_k-modules. Comparing the $L_{\mathfrak{gl}_n}(\lambda)$ components on each side of

$$\begin{aligned}\bigoplus_\lambda L_{\mathfrak{gl}_n}(\lambda) \otimes S_k^\lambda \cong V^{\otimes k} = V^{\otimes(k-1)} \otimes V &\cong \big(\bigoplus_\mu L_{\mathfrak{gl}_n}(\mu) \otimes S_{k-1}^\mu\big) \otimes V \\ &\cong \bigoplus_\mu \bigoplus_{\lambda/\mu=\square} L_{\mathfrak{gl}_n}(\lambda) \otimes S_{k-1}^\mu \cong \bigoplus_\lambda \Big(L_{\mathfrak{gl}_n}(\lambda) \otimes \big(\bigoplus_{\lambda/\mu} S_{k-1}^\mu\big)\Big)\end{aligned}$$

gives

$$S_k^\lambda \cong \bigoplus_{\lambda/\mu=\square} S_{k-1}^\mu.$$

Using the basis $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 1 \le i_1, \ldots, i_k \le n\}$ of $V^{\otimes k}$, the direct computation

$$\begin{aligned}\kappa(v_{i_1} \otimes \cdots\cdots \otimes v_{i_k}) &= \sum_{\ell=1}^k \big(v_{i_1} \otimes \cdots \otimes \sum_{i,j=1}^n E_{ij}E_{ji} v_{i\ell} \otimes \cdots \otimes v_{i_k}\big) \\ &\quad + \sum_{1\le\ell<m\le k} \sum_{i,j=1}^n \big(v_{i_1} \otimes \cdots \otimes E_{ji} v_{i_\ell} \otimes \cdots \otimes E_{ij} v_{i_m} \otimes \cdots \otimes v_{i_k} \\ &\qquad + v_{i_1} \otimes \cdots \otimes E_{ij} v_{i_\ell} \otimes \cdots \otimes E_{ji} v_{i_m} \otimes \cdots \otimes v_{i_k}\big) \\ &= \left(kn + 2\sum_{1\le\ell<m\le k} s_{\ell m}\right)(v_{i_1} \otimes \cdots \otimes v_{i_k}) = (kn + 2z_k)(v_{i_1} \otimes \cdots \otimes v_{i_k})\end{aligned}$$

Shows that

$$\kappa = kn + 2 \sum_{1\le \ell < m \le k} s_{\ell m}, \qquad \text{as operators on } V^{\otimes k}.$$

Since κ is a central element of $U\mathfrak{gl}_n$ and

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda)\le n}} L(\lambda) \otimes S_k^\lambda \qquad \text{as } (U\mathfrak{gl}_n, \mathbb{C}S_k) \text{ bimodules},$$

it follows from (???) that

$$z_k = \sum_{1\le \ell < m \le k} s_{\ell m} \qquad \text{acts on } S_k^\lambda \text{ by} \qquad \sum_{b\in\lambda} c(b).$$

Thus, in (???),

$$m_k = \sum_{1\le \ell < k} s_{\ell k} = z_k - z_{k-1} \qquad \text{acts on } S_{k-1}^\mu \text{ by the constant} \qquad c(\lambda/\mu).$$

Since the values $c(\lambda/\mu)$ are distinct for the distinct summands in (???),

$$S_{k-1}^\mu = \{ v \in S_k^\lambda \mid m_k v = c(\lambda/\mu) v\},$$

the $c(\lambda/\mu)$ eigenspace of m_k in S_k^λ. Iterating the decomposition (???) gives

$$S_k^\lambda = \bigoplus_{p\in \hat{S}_k^\lambda} S_1^{\Box},$$

and, since $S_1^{\Box}$ is one dimensional, this determines (up to constants) a unique

$$\text{basis of } S_k^\lambda \qquad \{v_p \mid p \in \hat{S}_k^\lambda\} \qquad \text{such that} \qquad m_i v_p = c(p(i)) v_p.$$

In $\mathbb{C}S_k$, $s_i m_i s_i + s_i = m_{i+1}$, and so

$$s_i m_i + 1 = m_{i+1} s_i \qquad \text{and} \qquad s_i m_j = m_j s_i, \quad \text{for } j \ne i, i+1.$$

Write $(s_i)_{pq}$ to denote the (p,q) entry of the matrix determined by the action of s_i on S_k^λ with respect to the basis in (???). Then

$$(s_i)_{pp}(m_i)_{pp} + 1 = (m_{i+1})_{pp}(s_i)_{pp} \qquad \text{giving} \qquad (s_i)_{pp} = \frac{1}{(m_{i+1})_{pp} - (m_i)_{pp}}.$$

Then .... COPY FROM NOTES. □

Corollary 1.2. *As $(U\mathfrak{gl}_n, \mathbb{C}S_k)$ bimodules,*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda)\le n}} L_{\mathfrak{gl}_n}(\lambda) \otimes S_k^\lambda,$$

where S_k^λ are simple S_k modules.

Corollary 1.3. *For $\lambda \in \hat{S}_k$, and $\mu \in \hat{S}_{k-1}$,*

$$\mathrm{Res}^{S_k}_{S_{k-1}}(S^\lambda_k) \cong \bigoplus_{\lambda/\nu=\square} S^\nu_{k-1} \qquad \text{and} \qquad \mathrm{Ind}^{S_k}_{S_{k-1}}(S^\mu_{k-1}) \cong \bigoplus_{\nu/\mu=\square} S^\nu_k. \tag{1.3}$$

where the first sum is over all partitions ν that are obtained from λ by removing a box, and the second sum is over all partitions ν which are obtained from μ by adding a box.

Corollary 1.4. *Let $(\mathbb{C}S_k)_{pq}$ be the (p,q) (simultaneous) eigenspace of $\mathbb{C}S_k$ with respect to the action of $m_1,\ldots,m_k$ by left and right multiplication,*

$$(\mathbb{C}S_k)_{pq} = \{a \in \mathbb{C}S_k \mid \text{for } 1 \le i,j \le k,\ \ m_i a = c(p(i))a \text{ and } am_j = c(q(j))m_j\}.$$

Then $\dim((\mathbb{C}S_k)_{pq}) = 1$ and there exist matrix units

$$e^\lambda_{pq}, \qquad \lambda \in \hat{S}_k, \quad p,q \in \hat{S}^\lambda_k$$

such that

$$(\mathbb{C}S_k)_{pq} = \mathbb{C}e_{pq} \qquad \text{and} \qquad e^\lambda_{pq} e^\mu_{rs} = \delta_{\lambda\mu}\delta_{qr} e^\lambda_{ps}.$$

Corollary 1.5. *Let $\Phi\colon \mathbb{C}S_k \to \mathrm{End}(V^{\otimes k})$ and $\Psi\colon U\mathfrak{gl}_n \to \mathrm{End}(V^{\otimes k})$ be the representations of S_k and $\mathfrak{gl}_n$ corresponding to their actions on $V^{\otimes k}$. Then*

$$\mathrm{End}_{GL_n(\mathbb{C})}(V^{\otimes k}) = \Phi(\mathbb{C}S_k) \qquad \text{and} \qquad \mathrm{End}_{\mathbb{C}S_k}(V^{\otimes k}) = \Psi(U\mathfrak{gl}_n),$$

and

$$\ker\Phi = \langle \sum_{w \in S_{n+1}} \det(w) w \rangle,$$

the ideal of $\mathbb{C}S_k$ generated by the alternating sum of the permutations in the subgroup S_n ($\ker\Phi = 0$ if $n \le k$).

1.1 The tower $\hat{B}$

If $\lambda \in \hat{O}_n$ then

$$L_{O_n}(\lambda) \otimes V \cong \bigoplus_{\substack{\mu/\lambda=\square \\ \text{or } \lambda/\mu=\square}} L_{O_n}(\mu), \qquad \text{as } O_n(\mathbb{C})\text{-modules},$$

where the sum is over $\mu \in \hat{O}_n$ that are obtained from λ by adding or removing a box. Build a graph $\hat{B}(n)$ which encodes the $O_n(\mathbb{C})$-module decomposition of $V^{\otimes k}$, $k \in \mathbb{Z}_{\ge 0}$, by setting

vertices on level k: $\qquad \hat{B}_k(n) = \{\lambda \in \hat{O}_n \mid k - |\lambda| \in 2\mathbb{Z}_{\ge 0}\}$, and
an edge $\lambda \to \mu$, if $\mu \in \hat{B}_{k+1}(n)$ is obtained from $\lambda \in \hat{B}_k(n)$ by adding or removing a box, (1.4)

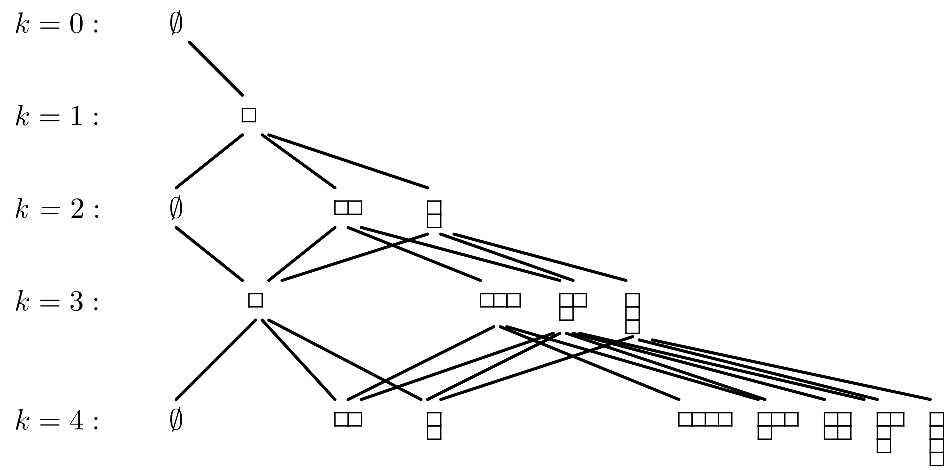
Theorem 1.6. *Define elements $m_1,\ldots,m_k \in \mathbb{C}B_k(n)$ by*

$$m_1 = 0, \qquad \text{and} \qquad m_i = \frac{k(n-1)}{4} + \sum_{\ell=1}^{i-1} s_{\ell i} - e_{\ell i}, \qquad \text{for } i > 1.$$

Then

(a) $m_i m_j = m_j m_i$ for $1 \le i,j \le n$.

(b) The eigenvalues of the elements m_i are given by the diagram



in the sense that if

$\hat{B}_k$ *is the set of vertices on level k, and*

$\hat{B}_k^\lambda = \{$*paths* $p = (\emptyset \to p^{(1)} \to p^{(2)} \to cdots \to p^{(k)} = \lambda)$ *to* λ *in* $\hat{B}\}$, *for* $\lambda \in \hat{B}_k$,

then

$$\hat{B}_k \text{ is an index set for the simple } \mathbb{C}B_k \text{ modules, } B_k^\lambda, \qquad \text{and}$$

$$B_k^\lambda \qquad \text{has a basis} \qquad \{v_p \mid p \in \hat{B}_k^\lambda\} \qquad \text{with} \qquad m_i v_p = c(p(i))v_p,$$

where

$$\begin{cases} c\big(p^{(i)}/p^{(i-1)}\big) + \frac{n-1}{2}, & \text{if } p^{(i)}/p^{(i-1)} = \square, \\ -c\big(p^{(i-1)}/p^{(i)}\big) - \frac{n-1}{2}, & \text{if } p^{(i-1)}/p^{(i)} = \square, \end{cases}$$

(c) $\kappa = m_k + m_{k-1} + \cdots + m_2$ *is a central element of* $\mathbb{C}B_k(n)$ *and*

$$\kappa \text{ acts on } B_k^\lambda \text{ by the constant} \qquad \frac{n-1}{2} + \sum_{b \in \lambda} c(b).$$

Proof. Let κ be the Casimir element of $\mathfrak{so}_n$ as in ???. Then

$$\begin{aligned} \kappa(v_{i_1} \otimes \cdots\cdots \otimes v_{i_k}) &= -\frac{1}{4}\Big(\sum_{\ell=1}^k v_{i_1} \otimes \cdots \otimes \sum_{i,j=1}^n (E_{ij} - E_{ji})^2 v_{i_\ell} \otimes \cdots \otimes v_{i_k}\Big) \\ &\quad - \frac{1}{4} 2 \sum_{1 \le \ell < m \le k} \sum_{i,j=1}^n v_{i_1} \otimes \cdots \otimes (E_{ij} - E_{ji})v_{i_\ell} \otimes \cdots \otimes (E_{ij} - E_{ji})v_{i_m} \otimes \cdots \otimes v_{i_k} \\ &= \left(-\frac{1}{4}\Big(\sum_{\ell=1}^k 1 - n - n + 1\Big) - \frac{1}{4} 2 \cdot 2 \sum_{1 \le \ell < m \le k} (e_{\ell m} - s_{\ell m}) \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}) \end{aligned}$$

since $(E_{ij} - E_{ji})^2 = E_{ij}^2 - E_{ij}E_{ji} - E_{ji}E_{ij} + E_{ji}^2$ and $E_{ij}^2 v_{i_\ell} = 0$ unless $i = j = i_\ell$. Thus, as operators on $V^{\otimes k}$,

$$\kappa = -\frac{1}{4}k(2 - 2n) + \sum_{1 \le \ell < m \le k} (s_{\ell m} - e_{\ell m}) = \frac{k(n-1)}{2} + \sum_{1 \le \ell < m \le k} s_{\ell m} - e_{\ell m}.$$

Since κ is a central element of $U\mathfrak{gl}_n$ and

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \le n}} L(\lambda) \otimes B_k^\lambda \qquad \text{as } (U\mathfrak{so}_n, \mathbb{C}B_k(n)) \text{ bimodules,}$$

it follows from (???) that

$$\frac{k(n-1)}{2} + \sum_{1 \le \ell < m \le k} s_{\ell m} - e_{\ell m} \qquad \text{acts on } B_k^\lambda \text{ by} \qquad (n-1)|\lambda| + \sum_{b \in \lambda} c(b).$$

The last statement follows since

$$m_1 + \cdots + m_k = \frac{k(n-1)}{2} + \sum_{1 \le \ell < m \le k} s_{\ell m} - e_{\ell m},$$

for every $k \in \mathbb{Z}_{>0}$. □

References

[HR] T. Halverson and A. Ram, *Partition algebras*, European J. Combinatorics **26** (2005), 869–921.

[Ko1] M. Kosuda, *Irreducible representations of the party algebra*, preprint 2004.

[Ko2] M. Kosuda, *Characterization of the party algebras* Ryukyu Math. J. **13** (2003), 199–228.

[Ta] K. Tanabe, *On the centralizer algebra of the unitary reflection group* $G(m,p,n)$, Nagoya Math. J. **148** (1997), 113-126.

[Dr1] V.G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.