Knizhnik-Zamolodchikov

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

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1 KZ connections

1.1 Connections

Let $\mathcal{O}_X$ be a commutative $\mathbb{C}$-algebra with 1. A *derivation* of $\mathcal{O}_X$ is a $\mathbb{C}$-linear map

$$\partial\colon \mathcal{O}_X \to \mathcal{O}_X \qquad \text{such that} \qquad \partial(fg) = \partial(f)g + f\partial(g),$$

for $f, g \in \mathcal{O}_X$. Let

$$\mathcal{I} = \ker\begin{pmatrix} \mathcal{O}_X \otimes \mathcal{O}_X & \to & \mathcal{O}_X \\ a \otimes a' & \mapsto & aa' \end{pmatrix} \qquad \text{and} \qquad \Omega^1_X = \mathcal{I}/\mathcal{I}^2.$$

Then $\mathcal{O}_X$ acts on Ω^1_X by

$$f\big(\sum g_i \otimes h_i\big) = \sum f g_i \otimes h_i = \sum g_i \otimes f h_i \mod \mathcal{I}^2,$$

for $f \in \mathcal{O}_X$ and $\sum g_i \otimes h_i \in \mathcal{I}$. Let $\mathrm{Der}(\mathcal{O}_X)$ be the vector space of derivations of $\mathcal{O}_X$. Then

$$\begin{array}{rccc} d\colon & \mathcal{O}_X & \longrightarrow & \Omega^1_X \\ & f & \longmapsto & f \otimes 1 - 1 \otimes f \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \xrightarrow{\sim} & \mathrm{Der}(\mathcal{O}_X) \\ \varphi & \longmapsto & fd \end{array}$$

are $\mathcal{O}_X$ module homomorphisms. If $\Omega^!_X$ is a reflexive $\mathcal{O}_X$ module then $\Omega^1_X = \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Der}(\mathcal{O}_X), \mathcal{O}_X)$.

Let M be a $\mathcal{O}_X$ module. A *connection* on M is a $\mathbb{C}$-linear map

$$\nabla\colon M \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} M \qquad \text{such that} \qquad \nabla(fm) = d(f) \otimes m + f\nabla(m),$$

for $f \in \mathcal{O}_X$ and $m \in M$. Let ∇ be a connection on M and define

$$\begin{array}{ccc} \mathrm{Der}(\mathcal{O}_X) & \longrightarrow & \mathrm{End}_{\mathbb{C}}(M) \\ \partial & \longmapsto & \nabla_\partial \end{array} \qquad \text{by} \qquad \nabla_\partial = (\partial \otimes \mathrm{id}_M) \circ \nabla,$$

so that

$$\nabla_\partial\colon M \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} M \xrightarrow{\partial\otimes \mathrm{id}_M} \mathcal{O}_X \otimes_{\mathcal{O}_X} M = M.$$

Then, for $f, g \in \mathcal{O}_X$, $\partial, \partial_1, \partial_1 \in \mathrm{Der}(\mathcal{O}_X)$ and $m \in M$,

(a) $\nabla_\partial(fm) = \partial(f)m + f\nabla_\partial(m)$,

(b) $\nabla_{f\partial_1 + g\partial_2}(m) = f\nabla_{\partial_1}(m) + g\nabla_{\partial_2}(m)$.

If Ω^1_X is a reflexive $\mathcal{O}_X$ module then the connection ∇ is determined by the map $\partial \mapsto \nabla_\partial$ with the properties (a) and (b).

1.2 Configuration space

The symmetric group S_k acts on $\mathbb{C}^k$ by permuting coordinates. Then S_k is a reflection group and

$$H_{\varepsilon_i-\varepsilon_j} = \{(z_1,\ldots,z_k)\in\mathbb{C}^k \mid z_i - z_j = 0\}, \qquad 1\le i<j\le k,$$

are the reflecting hyperplanes for the reflections in S_k. The *configuration spaces* are

$$\mathcal{D}_k(\mathbb{C}) = \mathbb{C}^k\backslash\Big(\bigcup_{1\le i<j\le k} H_{\varepsilon_i-\varepsilon_j}\Big) \qquad \text{and} \qquad \mathcal{C}_k(\mathbb{C}) = \mathcal{D}_k(\mathbb{C})/S_k.$$

The *braid group* is

$$\mathcal{B}_k = \pi_1(\mathcal{C}_k(\mathbb{C})) = \{\text{braids on } k \text{ strands}\}.$$

The *pure braid group* is

$$P\mathcal{B}_k = \pi_1(\mathcal{D}_m(\mathbb{C})) = \{\text{braids with } i\text{th top dot connected to the bottom } i\text{th dot}\},$$

with an exact sequence

$$\{1\} \longrightarrow P\mathcal{B}_k \longrightarrow \mathcal{B}_k \longrightarrow S_k \longrightarrow \{1\}.$$

1.3 Knizhnik-Zamolodchikov and the classical Yang-Baxter equation

Let $\mathfrak{g}$ be a Lie algebra and let $V_1,\ldots,V_k$ be $\mathfrak{g}$ modules. Let

$$\begin{array}{rccc} r\colon & \mathbb{C}\backslash\{0\} & \longrightarrow & \mathfrak{g}\otimes\mathfrak{g} \\ & z & \longmapsto & \sum r^{(1)}\otimes r^{(2)} \end{array}$$

and let

$$r_{ij}(z)(v_1\otimes\cdots v_k) = v_1\otimes\cdots v_{j-1}\otimes r^{(1)}v_j\otimes v_{j+1}\otimes\cdots\otimes v_{k-1}\otimes r^{(2)}v_k\otimes v_{k+1}\otimes\cdots\otimes v_m.$$

The *KZ-connection* (Knizhnik-Zamolodchikov connection) is

$$\nabla = \sum_{1\le i<j\le k} r_{ij}(z_i - z_j)(dz_i - dz_j),$$

a 1-form on $\mathcal{D}_k(\mathbb{C})$ with values in $\mathrm{End}(V_1\otimes\cdots\otimes V_k$. Then ∇ defines a connection on the trivial bundle over $\mathcal{D}_k(\mathbb{C})$ with fiber $V_1\otimes\cdots V_k$. The connection ∇ is flat if and only if r satisfies the *classical Yang-Baxter equation*:

$$[r_{12}(z_1-z_2), r_{23}(z_2-z_3)] + r_{12}(z_1-z_2), r_{13}(z_1-z_3)] + [r_{13}(z_1-z_3), r_{23}(z_2-z_3)] = 0.$$

The *KZ-equations* are

$$\frac{\partial f}{\partial z_j} = \sum_{\substack{k=1\\k\ne j}}^{m} r_{jk}(z_j-z_k)f, \qquad \text{for} \qquad f\colon \mathcal{D}_k(\mathbb{C}) \longrightarrow V_1\otimes\cdots V_k,$$

the conditions for f to be a covariant flat section of the bundle $\mathcal{D}_k(\mathbb{C})\times(V_1\otimes\cdots\otimes V_m)$ with connection ∇. The mondromy of this connection is a representation of the pure braid group $P\mathcal{B}_k$.

If $V_1 = V_2 = \cdots = V_k$ then S_k acts on $\mathcal{D}_k(\mathbb{C})\times(V_1\otimes\cdots\otimes V_k)$ by

$$w((z_1,\ldots,z_k), v_1\otimes\cdots v_k) = z_{w(1)},\ldots z_{w(k)}), v_{w(1)}\otimes\cdots\otimes v_{w(k)}),$$

and ∇ is S_k invariant. Thus ∇ defines a connection in a bundle over $\mathcal{C}_k(\mathbb{C})$ with fiber $V^{\otimes k}$. The monodromy of this connection is a representation of the braid group $\mathcal{B}_k$.

If $\langle , \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is a nondegenerate form and

$$t = \sum_i b_i \otimes b_i^*,$$

where $\{b_i\}$ is a basis of $\mathfrak{g}$ and $\{b_i^*\}$ is the dual basis of with respect to $\langle , \rangle$ then

$$r(z_1 - z_2) = \frac{1}{z_1 - z_2} t$$

satsfies the CYBE (classical Yang-Baxter equation).

1.4 Quantization and KZ

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Let $\mathfrak{g}_h$ be a Lie algebra over $\mathbb{C}[[h]]$ such that $\mathfrak{g}_h \cong \mathfrak{g}[[h]]$ as $\mathbb{C}[[h]]$ modules. Let

$$t_h \in S^2(\mathfrak{g}_h)^G.$$

There *exists* a quasitriangular quasiHopf algebra $A_{\mathfrak{g}_h, t_h}$ over $\mathbb{C}[[h]]$ such that

$$A_{\mathfrak{g}_h, t_h} \cong \widehat{U(\mathfrak{g}_h)}, \qquad \text{as a } \mathbb{C}[[h]] \text{ algebra.}$$

Proof. Let

$$A_h = \widehat{U(\mathfrak{g}_h)} \qquad \text{with} \qquad \mathcal{R}_{\mathrm{KZ}} = e^{ht_h/2}.$$

Note that $\Delta_h^{\mathrm{op}} = \mathcal{R}_{\mathrm{KZ}} \Delta_h \mathcal{R}_{\mathrm{KZ}}^{-1}$ by the $\mathfrak{g}_h$ invariance of t_h. The *universal KZ equation* is

$$\frac{\partial f}{\partial z_j} = \hbar \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{t_h^{jk}}{z_j - z_k} f, \qquad j = 1, 2, \ldots, m, \qquad (KZ_m)$$

where $\hbar = \frac{h}{2\pi i}$ and $f \colon \mathbb{C}^m \to (U\mathfrak{g}_h)^{\otimes m}$. Note that

$$\mathcal{R}_{\mathrm{KZ}} = e^{ht_h/2} \qquad \text{is the monodromy of } (KZ_2).$$

Define

$$\Phi_{KZ} = g_1^{-1} g_2 \qquad \text{in} \quad (U\mathfrak{g}_h)^{\otimes 3} \quad \text{by}$$

analytic solutions $g_1 \colon \mathbb{C} \to (U\mathfrak{g}_h)^{\otimes 3}$ and $g_2 \colon \mathbb{C} \to (U\mathfrak{g}_h)^{\otimes 3}$ of the equation

$$g'(x) = \hbar \left(\frac{t_h^{12}}{x} + \frac{t_h^{23}}{x - 1} \right) g(x),$$

so that

$$f(z_1, z_2; z_3) = (z_3 - z_1)^{h(t_h^{12} + t_h^{13} + t_h^{23})} g\left(\frac{z_2 - z_1}{z_3 - z_1} \right)$$

is a solution of (KZ_3) (using the invariance of (KZ_3) under transformations $z_i \mapsto a z_i + b$). Here g_1 and g_2 have asymptotic behaviour

$$\begin{aligned} g_1(x) &\sim x^{ht_h^{12}} && \text{as } x \to 0 \\ g_2(x) &\sim (1 - x)^{ht_h^{23}} && \text{as } x \to 1. \end{aligned}$$

Then $(\widetilde{U\mathfrak{g}[[h]]}, \Delta, \mathcal{R}_{\mathrm{KZ}}, \Phi_{\mathrm{KZ}})$ is a quasi Hopf algebra.

We want a Hopf algebra, i.e. a quasi Hopf algebra with $\Phi_h = 1 \otimes 1 \otimes 1$. The point is that one can *twist* to

$$(U_h\mathfrak{g}, \mathcal{R}_h, \Phi_h = \mathrm{id}), \qquad \text{where } U_h\mathfrak{g} \text{ is the quantum group,}$$

i.e. there exists $\mathcal{F} \in U\mathfrak{g}[[h]] \otimes U\mathfrak{g}[[h]]$ such that

$$\begin{aligned} \Delta_h &= \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \\ \Phi_h &= \mathcal{F}_{12}(\Delta \otimes \mathrm{id})(\mathcal{F})\Phi_{\mathrm{KZ}}(\mathrm{id} \otimes \Delta)(\mathcal{F})^{-1}\mathcal{F}_{23}^{-1}, \\ \mathcal{R}_h &= \mathcal{F}_{21}\mathcal{R}_{\mathrm{KZ}}\mathcal{F}_{12}^{-1}. \end{aligned}$$

□

1.5 Affine Lie algebras and KZ

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and fix a nondegenerate symmetric ad-invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$. The *affine Lie algebra* is the Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g}((z)) \oplus \mathbb{C}c$$

with c central and with

$$[f, g] = [f(z), g(z)] + \mathrm{res}_0\left(\langle f(z), \frac{dg}{dz}\rangle\right) \cdot c,$$

where $\mathrm{res}_0 h$ is the coefficient of z^{-1} in h. Let d be the derivation of $\hat{\mathfrak{g}}$ such that

$$[d, c] = 0 \qquad \text{and} \qquad [d, f] = z\frac{df}{dz}.$$

Let V be a finite dimensional irreducible $\mathfrak{g}$ module. The *loop representation* is the $\hat{\mathfrak{g}}$ module

$$V((z)) = V \otimes_{\mathbb{C}} \mathbb{C}((z)), \qquad \text{with } c \text{ acting by } 0.$$

A *level k highest weight representation of highest weight* λ is a $\hat{\mathfrak{g}}$ module W with a vector $w^+ \in W$ such that

(a) $W = (U\hat{\mathfrak{g}})w^+$,

(b) $cw^+ = \kappa w^+$,

(c) $z\mathfrak{g}[[z]]w^+ = 0$ and $\mathfrak{n}^+ w^+ = 0$,

(d) $hw^+ = \lambda(h)w^+$, for $h \in \mathfrak{h}$.

The *conformal weight* of W is

$$h_\kappa(\lambda) = \frac{\langle \lambda + \rho, \lambda + \rho\rangle - \langle \rho, \rho\rangle}{2(\kappa + h^\vee)}$$

Let W_0, W_1 be highest weight representations of level κ and highest weights $\lambda^{(0)}$ and $\lambda^{(1)}$. Let V be a simple $\mathfrak{g}$ module of highest weight μ. An *intertwining operator* is a $\hat{f}g$ module homomorphism

$$\mathcal{I}_V^{W_1,W_0} \colon W_1 \to W_0 \otimes V((z)) \qquad \text{and} \qquad \mathcal{J}_V^{W_1,W_0} = z^{h_\kappa(\lambda^{(0)}) + h_\kappa(\mu) - h_\kappa(\lambda^{(1)})}\mathcal{I}_I^{W_1,W_0}$$

is the *rescaled intertwining operator*. Let $W_0, \dots, W_k$ be highest weight $\hat{\mathfrak{g}}$ modules of level κ and let $V_1, \dots, V_k$ be simple $\mathfrak{g}$ modules. Let $F = \mathcal{I}_{V_1}^{W_0, W_1} \circ \cdots \circ \mathcal{I}_{V_k}^{W_{k-1}, W_k}$ so that

$$F \colon W_k \longrightarrow W_{k-1} \otimes V_k((z_k)) \longrightarrow W_{k-2} \otimes V_{k-1}((z_{k-1})) \otimes V_k((z_k)) \longrightarrow \cdots \longrightarrow W_0 \otimes V_1((z_1)) \otimes \cdots \otimes V_k((z_k)).$$

Then define

$$f \colon \mathcal{D}_k(\mathbb{C}) \longrightarrow V_1 \otimes \cdots V_k \qquad \text{by} \qquad f(z_1, \dots, z_m) = \langle w_0^+, F w_k^+ \rangle.$$

Then f satisfies the KZ equations for

$$r(z_1 - z_2) = \frac{1}{\kappa + h^\vee} \frac{t}{z_1 - z_2}.$$

1.6 Affine Lie algebras and quantum groups

Let

$$\tilde{\mathfrak{g}} = \mathfrak{g}[z, z^{-1}], \qquad \text{a subalgebra of } \hat{\mathfrak{g}}.$$

If V is a $\tilde{\mathfrak{g}}$ module then

$$V(1) \subseteq V(2) \subseteq \cdots \qquad \text{where} \qquad V(N) = \{ v \in V \mid (z\mathfrak{g})^N v = 0 \}.$$

The *smooth vectors* in V are the elements of

$$V(\infty) = \bigcup_{N \in \mathbb{Z}_{>0}} V(N).$$

The *category* $\mathcal{O}_\kappa$ is the category of $\tilde{\mathfrak{g}}$ modules V such that

(a) c acts by the scalar $\kappa - h$, where h is the Coxeter number,

(b) If $v \in V$ then $\dim((U\mathfrak{g})v)$ is finite,

(c) If $v \in V$ and $x \in z\mathfrak{g}[[z]]$ then $x^N v = 0$ for $N >> 0$,

(d) V is a finitely generated $\tilde{\mathfrak{g}}$ module.

Equivalently $\mathcal{O}_\kappa$ is the category of smooth $\tilde{\mathfrak{g}}$ modules such that

$$c \text{ acts by } \kappa - h \qquad \text{and} \qquad \dim(V(1)) \text{ is finite.}$$

Let $q = e^{-i\pi\kappa}$ and let $\tilde{\mathcal{O}}_\kappa$ be the category of finite dimensional $U_q\mathfrak{g}$ modules of type 1 which are U_0 semisimple.

Theorem 1.1. *There is an equivalence of categories*

$$\mathcal{O}_\kappa \cong \tilde{\mathcal{O}}_\kappa.$$

Let $V(\lambda)$ be the irreducible finite dimensional $\mathfrak{g}$ module of highest weight $\lambda \in P^+$. Extend $V(\lambda)$ to a $\mathfrak{g}[[z]]$ module by letting $z\mathfrak{g}[[z]]$ act trivially and let c act by $(\kappa - h)$. The *Weyl module* is the $\tilde{\mathfrak{g}}$ module given by

$$W^\kappa(\lambda) = U\tilde{\mathfrak{g}} \otimes_{U(\mathfrak{g}[[z]] \oplus \mathbb{C}c)} V(\lambda).$$

Let

$$L^\kappa(\lambda) \ \text{ be the unique simple quotient of } W^\kappa(\lambda).$$