Kazhdan-Lusztig polynomials

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1 Bar invariant bases

Proposition 1. *Let $(W, \leq)$ be a poset such that for $u, v \in W$ the interval $[u, v]$ is finite. Let M be the free $\mathbb{Z}[q, q^{-1}]$ modules with basis $\{T_w \mid w \in W\}$,*

$$M = \mathbb{Z}[q, q^{-1}]\text{-}span\{T_w \mid w \in W\},$$

and let $\bar{\ }: M \to M$ be a $\mathbb{Z}$-linear involution such that

$$\overline{q} = q^{-1} \qquad and \qquad \overline{T_w} = T_w + \sum_{v<w} a_{wv} T_v,$$

where $a_{vw} \in \mathbb{Z}[q, q^{-1}]$. Then

(a) There is a unique basis $\{C_w^- \mid w \in W\}$ such that

$$\overline{C_w} = C_w \qquad and \qquad C_w^- = T_w + \sum_{v<w} P_{vw}^- T_v \quad with\ P_{vw}^- \in q^{-1}\mathbb{Z}[q^{-1}].$$

(b) There is a unique basis $\{C_w^- \mid w \in W\}$ such that

$$\overline{C_w} = C_w \qquad and \qquad C_w^+ = T_w + \sum_{v<w} P_{vw}^+ T_v \quad with\ P_{vw}^+ \in q\mathbb{Z}[q].$$

Proof. (a) The p_{vw}^- are dertermined by induction:

$$P_{ww}^- = 1 \qquad \text{and} \qquad P_{uw}^- = \sum_{k \in \mathbb{Z}_{<0}} f_k q^k$$

where

$$f = \sum_{k \in \mathbb{Z}} f_k q^k = \sum_{u<z\leq w} a_{uz} \overline{P_{zw}^-} \quad (= P_{uw}^- - \overline{P_{uw}^-}).$$

(b) The P_{vw}^+ are dertermined by induction:

$$P_{ww}^+ = 1 \qquad \text{and} \qquad P_{uw}^+ = \sum_{k \in \mathbb{Z}_{>0}} f_k q^k$$

where

$$f = \sum_{k \in \mathbb{Z}} f_k q^k = \sum_{u<z\leq w} a_{uz} \overline{P_{zw}^+} \quad (= P_{uw}^+ - \overline{P_{uw}^+}).$$

□

The *dual module*

$$M^* = \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(M, \mathbb{Z}[q,q^{-1}])$$

is given a bar involution

$$\bar{\ }\colon M^* \to M^* \quad \text{defined by} \quad \langle \overline{\varphi}, m\rangle = \overline{\langle \varphi, \overline{m}\rangle}$$

If $\{T^w \mid w \in W\}$ is the dual basis to $\{T_w \mid w \in W\}$ then

$$\overline{T^w} = \sum_v b_{vw} T^v \qquad \text{where} \qquad b_{vw} = \langle \overline{T^w}, T_v\rangle = \overline{\langle T^w, \overline{T_v}\rangle} = \overline{\left\langle T^w, \sum_{z\le v} a_{zv} T_z \right\rangle}$$

so that $B = \overline{A^t}$. If $\{C^w \mid w \in W\}$ is the dual basis to $\{C_w \mid w \in W\}$ then

$$\overline{C^w} = C^w \qquad \text{since} \qquad \langle \overline{C^w}, C_v\rangle = \overline{\langle C^w, \overline{C_v}\rangle} = \overline{\langle C^w, C_v\rangle} = \delta_{vw},$$

and

$$C^w = \sum P^{vw} T^v \qquad \text{where} \qquad \delta_{vw} \quad \langle C^w, C_v\rangle = \left\langle \sum_u P^{uw} T_u, \sum_z P_{zv} T_z \right\rangle = \sum_u P^{uw} P_{uv}$$

so that

$$(P^{uw}) = \left((P_{uv})^{-1}\right)^t = (P^t)^{-1}.$$

2 The affine Hecke algebra

The *affine Hecke algebra* $\tilde{H}$ has $\mathbb{Z}[q,q^{-1}]$ basis $\{T_w \mid w \in \tilde{W}\}$,

$$\tilde{H} = \mathbb{Z}[q,q^{-1}]\text{-span}\{T_w \mid w \in \tilde{W}\}$$

with relations

$$\begin{array}{ll} T_{w_1}T_{w_2} = T_{w_1w_2}, & \text{if } \ell(w_1w_2) = \ell(w_1) + \ell(w_2), \\ T_{s_i}T_w = (q-q^{-1})T_w + T_{s_iw}, & \quad \text{if } \ell(s_iw) < \ell(w), \qquad (0 \le i \le n). \end{array}$$

The algebra $\tilde{H}$ also has bases

$$\{X^\lambda T_w \mid w \in W, \lambda \in P\} \qquad \text{and} \qquad \{T_v X^\mu \mid v \in W, \mu \in P\},$$

where

$$X^\lambda = T_{t_\lambda}, \ \ \text{if } \lambda \in P^+, \qquad \text{and} \qquad X^\lambda = X^\mu (X^\nu)^{-1},$$

if $\lambda = \mu - \nu$ with $\mu, \nu \in P^+$.

The *bar involution* on $\tilde{H}$ is the $\mathbb{Z}$-linear map $\bar{\ }\colon \tilde{H} \to \tilde{H}$ given by

$$\overline{q} = q^{-1} \qquad \text{and} \qquad \overline{T_w} = T_{w^{-1}}^{-1} \quad \text{for } w \in \tilde{W}.$$

Define elements $\mathbf{1}_0, \varepsilon_0 \in H$ by

$$\begin{array}{lll} \mathbf{1}_0^2 = \mathbf{1}_0, & \text{and} & T_{s_i}\mathbf{1}_0 = q\mathbf{1}_0, \ \ \text{for } 1 \le i \le n, \\ \varepsilon_0^2 = \varepsilon_0, & \text{and} & T_{s_i}\varepsilon_0 = q^{-1}\varepsilon_0, \ \ \text{for } 1 \le i \le n, \end{array}$$

and let

$$A_\mu = \varepsilon_0 X^\mu \mathbf{1}_0, \qquad \text{for } \mu \in P.$$

Proposition 2.

(a) $\overline{X^\lambda} = T_{w_0} X^{w_0\lambda} T_{w_0}^{-1}$, *for* $\lambda \in P$,

(b) $\overline{\mathbf{1}_0} = \mathbf{1}_0$ *and* $\overline{\varepsilon_0} = \varepsilon_0$.

(c) *If* $z \in \mathbb{Z}[P]^W$ *then* $\overline{z} = z$.

(d) $\overline{q^{-\ell(w_0)} A_{\lambda+\rho}} = q^{-\ell(w_0)} A_{\lambda+\rho}$.

The τ*-operators* are given by

$$\tau_i = T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}}.$$

Then

(a) $X^\lambda \tau_i = \tau_i X^{s_i\lambda}$,

(b) $\tau_i^2 = \dfrac{(q - q^{-1}X^{\alpha_i})(q - q^{-1}X^{-\alpha_i})}{(1 - X^{\alpha_i})(1 - X^{-\alpha_i})}$

(c) *displaystyle*$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}\text{ factors}}$

The *shift operator* is

$$\Delta = \prod_{\alpha\in R^+} (qX^{\alpha/2} - q^{-1}X^{-\alpha/2}).$$

Then

$$(T_i + q)\Delta = (s_i\Delta)(T_i - q) \qquad \text{and} \qquad \Delta\mathbb{C}[P]^W = \{h \in \tilde{H} \mid (t_i + q^{-1}h = 0 \text{ for } 1 \le i \le n\}$$

$$\varepsilon_0 E_\lambda = \mathbf{\Delta 1}_0 E_{\lambda-\rho} \qquad \text{and} \qquad \langle \Delta f, \Delta g\rangle_k = q^{Nk}\langle f, g\rangle_{k+1}.$$

The *??-trace* on $\tilde{H}$ is the linear map $\tau\colon \tilde{H} \to \mathbb{C}$ given by

$$\mathrm{tr}(h) = h|_1, \qquad \text{or, more precisely,} \quad \mathrm{tr}(T_w) = \delta_{w1}.$$

Define an inner product on $\tilde{H}$ by

$$\langle h_1, h_2\rangle = \mathrm{tr}(h_1h_2),$$

so that

$$\langle T_u, T_v\rangle = [T_{u^{-1}}T_v]_1 \qquad \text{and} \qquad \langle T_u, T_v\rangle = q^{\ell(u)??}\delta_{uv^{-1}}.$$

The *generic degrees* are $d_\lambda(q)$ given by

$$\mathrm{tr} = \sum_{\lambda\in\hat{H}} d_\lambda(q)\chi_H^\lambda.$$

The *Kazhdan-Lusztig* basis is defined by

$$\{h \in H \mid \langle h, h^\#\rangle \in 1 + q^{-1}\mathbb{Z}[q^{-1}], h = \overline{h}\}$$

or by the usual bar invariance and triangularity conditions.

3 Kazhdan-Lusztig polynomials

The *Iwahori-Hecke* algebra is the algebra over $\mathbb{Z}[q]$ given by generators T_w, $w \in W$ and relations

$$T_{s_i}T_w = \begin{cases} T_{s_iw}, & \text{if } s_iw > w, \\ qT_{s_iw} + (q-1)T_w, & \text{if } s_iw < w. \end{cases}$$

The *bar involution* on H is the $\mathbb{Z}$-algebra involution given by

$$\overline{q} = q^{-1} \qquad \text{and} \qquad \overline{T_w} = T_{w^{-1}}^{-1},$$

for $w \in W$. The *Kazhdan-Lusztig basis* of H is the basis $\{C_w \mid w \in W\}$ given by

(a) $\overline{C_w} = C_w$, and

(b) $C_w = T_w + \sum_{v \le w} p_{vw}(q) T_v$, where $p_{vw}(q) \in q\mathbb{Z}[q]$.

Kazhdan-Lusztig polynomials

(1) $P_{ww}(q) = 1$,

(2) $P_{xw}(q) = 0$, if $x \not< w$,

(3) $\deg(P_{xw}(q)) \le \frac{1}{2}(\ell(w) - \ell(x) - 1)$, if $x \ne w$.

Define

$$\mu(x, w) = \text{coefficient of the highest degree term in } P_{xw}(q),$$

which is the term of degree $\frac{1}{2}(\ell(w) - \ell(x) - 1)$. Then, if $sw < w$

$$P_{xw}(q) = \begin{cases} P_{sx,w}(q), & \text{if } sx > x, \\ P_{sx,sw}(q) + qP_{x,sw} - \sum_{sz<z} q^{\frac{1}{2}(\ell(w)-\ell(z))}\mu(z, sw)P_{x,z}, & \text{if } sx < x. \end{cases}$$

The W-*graph* has

Vertices: W

Edges: $x \leftrightarrow y$ if $\mu[x, y] = \begin{cases} \mu(x, y), & \text{if } x < y, \\ \mu(y, x), & \text{if } y < x, \end{cases}$

Then

$$KL(s)_{xx} = \begin{cases} -1, & \text{if } sx < x, \\ 1, & \text{if } sx > x, \end{cases}$$

$$KL(s)_{xy} = \begin{cases} \mu[x, y], & \text{if } sx < x,\ sy > y \text{ and } x \leftrightarrow y, \\ 0, & \text{otherwise}, \end{cases}$$

Define a relation $\le_L$ by taking the closure of the relation

$$x \le_L y \qquad \text{if} \qquad D_\ell(x) \not\subseteq D_\ell(y) \text{ and } x \leftrightarrow y \text{ is an edge.}$$

and define

$$x =_L y \quad \text{if } x \le_L y \text{ and } y \le_L x.$$

3.1 The case of dihedral groups

In type A_1,

$$H = \mathrm{span}\{1, T_1\} \qquad \text{with} \qquad T_1^2 = (q-1)T_1 + q.$$

So

$$\overline{T_1} = T_1^{-1} = (q^{-1}-1) + q^{-1}T_1, \qquad \text{and} \qquad C_1 = q^{-\frac{1}{2}}(1+T_1),$$

since

$$\overline{q^{-\frac{1}{2}}(1+T_1)} = q^{\frac{1}{2}}(1+T_1^{-1}) = q^{\frac{1}{2}}(1+q^{-1}T_1+(q^{-1}-1)) = q^{\frac{1}{2}}q^{-1}(1+T_1) = q^{-\frac{1}{2}}(1+T_1).$$

In type A_2, $H = \mathrm{span}\{1, T_1, T_2, T_1T_2, T_2T_1, T_1T_2T_2\}$ and

$$\begin{aligned}
C_1 &= q^{-\frac{1}{2}}(1+T_1), \\
C_2 &= q^{-\frac{1}{2}}(1+T_2), \\
C_1C_2 &= q^{-1}(1+T_1+T_2+T_1T_2) = C_{12}, \\
C_2C_1 &= q^{-1}(1+T_1+T_2+T_2T_1) = C_{21}, \\
C_1C_{21} &= q^{-\frac{3}{2}}(T_1T_2T_1 + T_1T_2 + (q-1)T_1 + q + T_1 + T_2T_1 + T_1 + T_2 + 1), \\
&= q^{-\frac{3}{2}}(T_1T_2T_1 + T_1T_2 + T_2T_1 + T_1 + T_2 + 1) + C_1,
\end{aligned}$$

so that

$$C_{121} = C_1C_{12} - C_1 = q^{-\frac{3}{2}}(T_1T_2T_1 + T_1T_2 + T_2T_1 + T_1 + T_2 + 1).$$

Note that $C_1^2 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C_1$. Then, using that $T_i = q^{\frac{1}{2}}C_i - 1$, to produce the matrices for the regular representation in the KL-basis,

$$\rho(T_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}} & q & q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} & q & 0 \\ 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & q \end{pmatrix} \qquad \text{and} \qquad \rho(T_2) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & q & 0 & 0 & 0 \\ q^{\frac{1}{2}} & 0 & 0 & q & q^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & q \end{pmatrix}$$

with rows and columns indexed by $1, C_1, C_{21}, C_2, C_{12}, C_{121}$.

In type B_2, $H = \mathrm{span}\{1, T_1, T_2, T_1T_2, T_2T_1, T_1T_2T_1, T_2T_1T_2, T_1T_2T_1T_2\}$, and

$$C_1C_2 = C_{12}, \quad C_2C_1 = C_{21}, \quad C_1C_{21} = C_{121} + C_1, \quad C_2C_{12} = C_{212} + C_2, \quad C_2C_{121} = C_{2121} + C_{21}.$$

where

$$\begin{aligned}
C_1 &= q^{-\frac{1}{2}}(1+T_1), \\
C_2 &= q^{-\frac{1}{2}}(1+T_2), \\
C_{12} &= q^{-1}(1+T_1+T_2+T_1T_2), \\
C_{21} &= q^{-1}(1+T_1+T_2+T_2T_1), \\
C_{121} &= q^{-\frac{3}{2}}(1+T_1+T_2+T_1T_2+T_2T_1+T_1T_2T_1), \\
C_{212} &= q^{-\frac{3}{2}}(1+T_1+T_2+T_1T_2+T_2T_1+T_2T_1T_2), \\
C_{1212} &= q^{-2}(1+T_1+T_2+T_1T_2+T_2T_1+T_1T_2T_1+T_2T_1T_2+T_2T_1T_2T_1).
\end{aligned}$$

and the matrices of the regular representation in the KL-basis are

$$\rho(T_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^{\frac12} & q & q^{\frac12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{\frac12} & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{\frac12} & q & q^{\frac12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac12} & q \end{pmatrix}$$

$$\rho(T_2) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{\frac12} & q & q^{\frac12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ q^{\frac12} & 0 & 0 & 0 & q & q^{\frac12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & q^{\frac12} & q & 0 \\ 0 & 0 & 0 & q^{\frac12} & 0 & 0 & 0 & q \end{pmatrix}$$

with rows and columns indexed by $1, C_1, C_{21}, C_{121}, C_2, C_{12}, C_{212}, C_{1212}$.

Proposition 3. *Let W be the dihedral group of order $2m$. Then*

$$C_w = q^{-\frac12 \ell(w)}\Big(\sum_{v\le w} T_v\Big), \qquad \text{so that} \qquad p_{vw}(q) = 1, \quad \text{for all } v \le w.$$

Proof. Let C_w be defined by the formula in the statement of the Theorem. If $s_1 w > w$ so that $w = s_2 s_1 s_2 s_1 \cdots$ then

$$\begin{aligned} C_{s_1} C_w &= q^{-\ell(w)/2} q^{-1/2}\left(\sum_{v\le w} T_v + \sum_{\substack{v\le s_1 w\\ s_1 v<v}} T_v + (q-1)\sum_{\substack{v<w\\ s_1 v<v}} T_v + q\sum_{\substack{v<w\\ s_2 v<v}} T_v\right) \\ &= q^{-\ell(w)/2} q^{-1/2}\left(\sum_{\substack{v\le s_1 w\\ s_2 v<v}} T_v + \sum_{\substack{v<w\\ s_1 v<v}} T_v + \sum_{\substack{v\le s_1 w\\ s_1 v<v}} T_v - \sum_{\substack{v<w\\ s_1 v<v}} T_v + q\sum_{v\le s_2 w} T_v\right) \\ &= C_{s_1 v} + q^{-\ell(w)/2} q^{1/2}\left(\sum_{v\le s_2 w} T_v\right) \\ &= C_{s_1 w} + C_{s_2 w}, \end{aligned}$$

and, if $s_1 w < w$ so that $w = s_1 s_2 s_1 s_2 \cdots$ then let $w' = s_1 w$ and $w'' = s_2 s_1 w$ so that

$$\begin{aligned} C_{s_1} C_w &= C_{s_1} C_{s_1 w'} = C_{s_1}\big(C_{s_1} C_{w'} - C_{s_2 w'}\big) \\ &= C_{s_1}\big(C_{s_1} C_{w'} - C_{w''}\big) = (q^{1/2} + q^{-1/2}) C_{s_1} C_{w'} - C_{s_1} C_{w''} \\ &= (q^{1/2} + q^{-1/2}) C_{s_1} C_{w'} - (q^{1/2} + q^{-1/2}) C_{w''}, \qquad \text{by induction,} \\ &= (q^{1/2} + q^{-1/2})(C_{s_1} C_{w'} - C_{w''}) = (q^{1/2} + q^{-1/2}) C_w. \end{aligned}$$

So,

$$C_{s_1}C_w = \begin{cases} C_{s_1w} + C_{s_2w}, & \text{if } s_1w > w, \text{ i.e. } w = s_2s_1s_2s_1\cdots, \\ (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C_w, & \text{if } s_1w < w, \text{ i.e. } w = s_1s_2s_1s_2\cdots. \end{cases} \tag{3.1}$$

In the first case, $\ell(s_2w) < \ell(w)$ and so, by induction, $C_{s_1w} = C_{s_1}C_w - C_{s_2w}$ is bar invariant. $\square$

From equation (???)

$$T_1C_w = \begin{cases} qC_w, & \text{if } s_1w < w, \text{ i.e. } w = s_1s_2s_1s_2\cdots, \\ q^{\frac{1}{2}}C_{s_1w} - C_w + q^{\frac{1}{2}}C_{s_2w}, & \text{if } s_1w > w, \text{ i.e. } w = s_2s_1s_2s_1\cdots. \end{cases}$$

For example, in the case $I_2(5)$,

$$\begin{array}{llll} C_2C_1 = C_{21}, & C_1C_{21} = C_{121} + C_1, & C_2C_{121} = C_{2121} + C_{21}, & C_1C_{2121} = C_{12121} + C_{121}, \\ C_1C_2 = C_{12}, & C_2C_{12} = C_{212} + C_2, & C_1C_{212} = C_{1212} + C_{12}, & C_2C_{1212} = C_{21212} + C_{212}, \end{array}$$

and the matrices of the regular representation in the KL-basis are

$$\rho(T_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}} & q & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{\frac{1}{2}} & q & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & q & q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & q & 0 \\ 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

$$\rho(T_2) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & q & q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} & q & 0 & 0 & 0 & 0 & 0 \\ q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & q & q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & q & q^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{2}} & q \end{pmatrix}$$

References

[Cu1] Curtis, C. "Representations of Hecke algebras." *Astérisque* **9** (1988): 13-60.