Kazhdan-Lusztig polynomials

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1 Kazhdan-Lusztig polynomials

The *Iwahori-Hecke* algebra is the algebra over $\mathbb{Z}[q]$ given by generators T_w, $w \in W$ and relations

$$T_{s_i}T_w = \begin{cases} T_{s_iw}, & \text{if } s_iw > w, \\ qT_{s_iw} + (q-1)T_w, & \text{if } s_iw < w. \end{cases}$$

The *bar involution* on H is the $\mathbb{Z}$-algebra involution given by

$$\overline{q} = q^{-1} \qquad \text{and} \qquad \overline{T_w} = T_{w^{-1}}^{-1},$$

for $w \in W$. The *Kazhdan-Lusztig basis* of H is the basis $\{C_w \mid w \in W\}$ given by

(a) $\overline{C_w} = C_w$, and

(b) $C_w = T_w + \sum_{v \le w} p_{vw}(q)T_v$, where $p_{vw}(q) \in q\mathbb{Z}[q]$.

Proposition 1. *Let W be the dihedral group of order $2m$. For all $v \le w$,*

$$p_{vw}(q) = 1.$$

Proof. We will show that

$$C_w = q^{-\ell(w)/2}\left(\sum_{v \le w} T_v\right).$$

If $s_1w > w$ so that $w = s_2s_1s_2s_1\cdots$ then

$$\begin{aligned}
C_{s_1}C_w &= q^{-\ell(w)/2}q^{-1/2}\left(\sum_{v\le w} T_v + sum_{v \le s_1w, s_1v<v}T_v + (q-1)\sum_{v<w, s_1v<v} T_v + q\sum_{v<w, s_2v<v} T_v\right) \\
&= q^{-\ell(w)/2}q^{-1/2}\left(\sum_{v\le s_1w, s_2v<v} T_v + \sum_{v<w, s_1v<v} T_v + \sum_{v \le s_w, s_1v<v} T_v - \sum_{v<w, s_1v<v} T_v + q\sum_{v\le s_2w} T_v\right) \\
&= C_{s_1v} + q^{-\ell(w)/2}q^{1/2}\left(\sum_{v\le s_2w} T_v\right) \\
&= C_{s_1w} + C_{s_2w},
\end{aligned}$$

and, if $s_1 w < w$ so that $w = s_1 s_2 s_1 s_2 \cdots$ then let $w' = s_1 w$ and $w'' = s_2 s_1 w$ so that

$$\begin{aligned} C_{s_1} C_w &= C_{s_1} C_{s_1 w'} = C_{s_1} \big(C_{s_1} C_{w'} - C_{s_2 w'} \big) \\ &= C_{s_1} \big(C_{s_1} C_{w'} - C_{w''} \big) = (q^{1/2} + q^{-1/2}) C_{s_1} C_{w'} - C_{s_1} C_{w''} \\ &= (q^{1/2} + q^{-1/2}) C_{s_1} C_{w'} - (q^{1/2} + q^{-1/2}) C_{w''}, \qquad \text{by induction,} \\ &= (q^{1/2} + q^{-1/2}) (c_{s_1} C_{w'} - C_{w''}) = (q^{1/2} + q^{-1/2}) C_w. \end{aligned}$$

In the first case, $\ell(s_2 w) < \ell(w)$ and so, by induction, $C_{s_1 w} = C_{s_1} C_w - C_{s_2 w}$ is bar invariant. □

References

[Cu1] Curtis, C. "Representations of Hecke algebras." *Astérisque* **9** (1988): 13-60.