Hopf algebras

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1 Hopf algebras

Let $\mathbb{K}$ be a commutative ring. A *vector space over* $\mathbb{K}$ is a free $\mathbb{K}$-module. Unless otherwise specified all maps between vector spaces over $\mathbb{K}$ are assumed to be $\mathbb{K}$-linear and, if V is a vector space over $\mathbb{K}$, then $\mathrm{id}\colon V \to V$ denotes the identity map from V to V.

An *algebra over* $\mathbb{K}$ is a vector space over $\mathbb{K}$ with a *multiplication* and an *identity element* $1 \in A$ such that

(a) m is *associative*, i.e. $(a_1a_2)a_3 = a_1(a_2a_3)$, for all $a_1, a_2, a_3 \in A$, and

(b) $1a = a1 = a$, for all $a \in A$.

Equivalently, an *algebra over* $\mathbb{K}$ is a vector space A over $\mathbb{K}$ with a *multiplication* $m\colon A \otimes A \to A$ and a *unit* $\iota\colon \mathbb{K} \to A$ such that

(a) m is *associative*, i.e. $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$, and

(b) (unit condition) $m(\iota \otimes \mathrm{id}) = m(\mathrm{id} \otimes \iota) = \mathrm{id}$.

The relationship between the identity $1 \in A$ and the unit $\iota\colon \mathbb{K} \to A$ is $\iota(1) = 1$.

Let A be an algebra over $\mathbb{K}$. An *A-module* is a vector space M over $\mathbb{K}$ with an A-action

$$\begin{array}{ccc} A \otimes M & \longrightarrow & M \\ a \otimes m & \longmapsto & am \end{array} \qquad \text{such that} \qquad (a_1a_2)m = a_1(a_2m), \quad \text{and} \quad 1m = m,$$

for all $a_1, a_2 \in A$ and $m \in M$.

Let M and N be A-modules. An *A-module morphism from M to N* is a map $\varphi\colon M \to N$ such that

$$\varphi(am) = a\varphi(m), \quad \text{for all } a \in A \text{ and } m \in M.$$

The set of A-module morphisms from M to N is denoted $\mathrm{Hom}_A(M, N)$.

A *Hopf algebra* is a vector space A over $\mathbb{K}$ with

$$\begin{array}{ll} \text{a multiplication,} & m\colon A \otimes A \longrightarrow A, \\ \text{a comultiplication,} & \Delta\colon A \longrightarrow A \otimes A, \\ \text{a unit,} & \iota\colon \mathbb{K} \longrightarrow A, \\ \text{a counit,} & \varepsilon\colon A \longrightarrow \mathbb{K}, \\ \text{an antipode,} & S\colon A \to A, \end{array}$$

such that

(1) m is associative, $\quad m(\mathrm{id}\otimes m) = m(m\otimes \mathrm{id})$,

(2) Δ is coassociative, $\quad (\mathrm{id}\otimes\Delta)\Delta = (\Delta\otimes\mathrm{id})\Delta$,

(3) (unit condition), $\quad m(\mathrm{id}\otimes\iota) = m(\iota\otimes\mathrm{id}) = \mathrm{id}$,

(4) (counit condition), $\quad (\mathrm{id}\otimes\varepsilon)\Delta = (\varepsilon\otimes\mathrm{id})\Delta = \mathrm{id}$,

(5) Δ is an algebra homomorphism, $\quad \Delta m = (m\otimes m)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta\otimes\Delta)$,

(6) ε is an algebra homomorphism, $\quad \varepsilon m = \varepsilon\otimes\varepsilon$,

(7) (antipode condition), $\quad \mu(id\otimes S)\Delta = \mu(S\otimes\mathrm{id})\Delta = 1\varepsilon$.

In condition (5) the algebra structure on $A\otimes A$ is given by

$$(a\otimes b)(c\otimes d) = ac\otimes bd, \quad \text{for } a,b,c,d\in A, \qquad \text{and} \qquad \begin{array}{rccc} \tau\colon & A\otimes A & \longrightarrow & A\otimes A \\ & a_1\otimes a_2 & \longmapsto & a_2\otimes a_1 \end{array}$$

In condition (6) we have identified the vector space $\mathbb{K}\otimes\mathbb{K}$ with $\mathbb{K}$. Since

???

the antipode $S\colon A\to A$ is an antihomomorphism

$$S(a_1a_2) = S(a_2)S(a_1), \qquad \text{for all } a_1,a_2\in A.$$

Let A be a Hopf algebra over $\mathbb{K}$. If $a\in A$ write

$$\Delta(a) = \sum_a a_{(1)}\otimes a_{(2)}$$

to express $\Delta(a)$ as an element of $A\otimes A$. This notation is called *Sweedler notation* and is a standard notation for working with Hopf algebras. It should be bothersome, it is simply a way to write $\Delta(a)$ so that it looks like an element of $A\otimes A$ without having to go through the rigmarole of actually choosing a basis in A.

For A-modules M, N and P, define the *tensor product* to be the A-module $M\otimes N$ with A-action given by

$$a(m\otimes n) = \Delta(a)(m\otimes n) = \sum_a a_{(1)}m\otimes a_{(2)}n, \qquad \text{if} \quad \Delta(a) = \sum_a a_{(1)}\otimes a_{(2)},$$

the *trivial module* $\mathbf{1} = \mathbb{K}\cdot 1$, with A-action given by

$$a\cdot 1 = \varepsilon(a)\cdot 1,$$

and the *dual module* $M^* = \mathrm{Hom}_{\mathbb{K}}(M,\mathbb{K})$ with A-action given by

$$(a\varphi)(m) = \varphi(S(a)m), \qquad \text{for } \varphi\in M^*,\ m\in m,$$

The definition of a Hopf algebra is exactly designed sot that $M\otimes N$, $\mathbb{K}_\varepsilon$ and M^* are well defined A-modules and the maps

$$\begin{array}{ccc} (M\otimes N)\otimes P & \longrightarrow & M\otimes(N\otimes P) \\ m\otimes n\otimes p & \longmapsto & m\otimes n\otimes p \end{array}$$

$$\begin{array}{rcl} M\otimes \mathbf{1} & \longrightarrow & M \\ m\otimes \mathbf{1} & \longmapsto & m \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbf{1}\otimes M & \longrightarrow & M \\ \mathbf{1}\otimes m & \longmapsto & m \end{array}$$

and

$$\begin{array}{rcl} M^*\otimes M & \longrightarrow & \mathbf{1} \\ m\otimes \varphi & \longmapsto & \varphi(m) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbf{1} & \longrightarrow & M\otimes M^* \\ 1 & \longmapsto & \sum_i e_i\otimes e^i \end{array} \tag{1.1}$$

are A-module homomorphisms. The sum in (???) is over a $\mathbb{K}$-basis $\{e_i\}$ of M and we only consider this map when this sum exists. CHECK on and REMARK on the order of M and M^* in the tensor products.

Let A be a Hopf algebra. The vector space A is an A-module where the action of A on A is given by

$$\begin{array}{rcl} A\otimes A & \longrightarrow & A \\ a\otimes b & \longmapsto & \sum_a a(1)bS(a_{(2)}), \end{array} \qquad \text{where} \qquad \Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}.$$

The linear transformation of A determined by the action of an element $a\in A$ is denoted ad_a. Thus,

$$\mathrm{ad}_a(b)=\sum_a a_{(1)}bS(a_{(2)}), \qquad \text{for all } b\in A.$$

Let M be an A-module and let $\rho\colon A\to \mathrm{End}(M)$ be the corresponding representation of A, i.e. the map

$$\rho\colon \quad \begin{array}{rcl} A & \longrightarrow & \mathrm{End}(M) \\ a & \longmapsto & \rho(a) \end{array}$$

where $\rho(a)$ is the linear transformation of M determined by the action of a. Note that $\mathrm{End}(M)\cong M\otimes M^*$ as a vector space. On the other hand $M\otimes M^*$ is an A-module. The definition of the adjoint action is exactly designed so that the composite map

$$\rho\colon A\to \mathrm{End}(M)\cong M\otimes M^* \qquad \text{is an } A\text{-module homomorphism.}$$

2 Quasitriangular Hopf algebras

Let $A=(A,m,\Delta,\varepsilon,\iota,S)$ be a Hopf algebra and let τ be the $\mathbb{K}$ linear map

$$\tau\colon \quad \begin{array}{rcl} A\otimes A & \longrightarrow & A\otimes A \\ a\otimes b & \longmapsto & b\otimes a \end{array}. \qquad \text{Let} \qquad \Delta^{\mathrm{op}}=\tau\Delta$$

so that, if $a\in A$ and

$$\Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}, \qquad \text{then} \qquad \Delta^{\mathrm{op}}(a)=\sum_a a_{(2)}\otimes a_{(1)}. \tag{2.1}$$

Then $(A,m,\Delta^{\mathrm{op}},\iota,\varepsilon,S^{-1})$ is a Hopf algebra.

The map $\tau\colon A\otimes A\to A\otimes A$ is an algebra automorphism of $A\otimes A$ (the algebra structure on $A\otimes A$ is as gievn in (???)) and the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A\otimes A \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \tau} \\ A & \xrightarrow{\Delta^{\mathrm{op}}} & A\otimes A \end{array}$$

Sometimes we are lucky and we can replace τ by an *inner* automorphism.

Let U be a Hopf algebra with an invertible element

$$\mathcal{R} \in U \otimes U \qquad \text{such that} \qquad \mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a), \tag{2.2}$$

for all $a \in U$. The pair $(U, \mathcal{R})$ is a *quasitriangular Hopf algebra* if $\mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a)$, for all $a \in U$ and

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23} \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}. \tag{2.3}$$

where, if $\mathcal{R} = \sum b_i \otimes b^i$, then

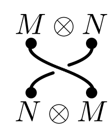
$$\mathcal{R}_{12} = \sum b_i \otimes b^i \otimes 1, \quad \mathcal{R}_{13} = \sum b_i \otimes 1 \otimes b^i, \quad \text{and} \quad \mathcal{R}_{23} = \sum 1 \otimes b_i \otimes b^i.$$

The identities in (???) relate the $\mathcal{R}$-matrix to coproduct and the relations between the $\mathcal{R}$ matrix and the counit and antipode are given by

$$\begin{gathered}(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = 1 = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}), \\ (S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1} \qquad \text{and} \qquad (S \otimes S)(\mathcal{R}) = \mathcal{R}.\end{gathered} \tag{2.4}$$

For any two U modules M and N, the map

$$\begin{aligned}\check{R}_{MN}\colon \quad M \otimes N &\longrightarrow \quad N \otimes M \\ m \otimes n &\longmapsto \sum b_i n \otimes a_i m\end{aligned}$$

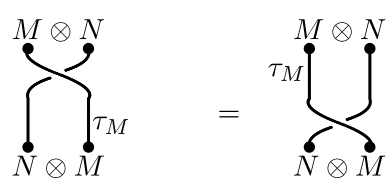


is a U module isomorphism since

$$\begin{aligned}\check{R}_{MN}(a(m \otimes n)) &= \check{R}_{MN}(\Delta(a)(m \otimes n)) = \sigma R\Delta(a)(m \otimes n) \\ &= \sigma\Delta^{op}(a)\sigma\sigma^{-1}R(m \otimes n) = \Delta(a)\check{R}_{MN}(m \otimes n)\end{aligned} \tag{2.5}$$

In order to be consistent with the graphical calculus the operators $\check{R}_{MN}$ should be written *on the right.*

For $U_h\mathfrak{g}$ modules M and N and a U-module isomorphism $\tau_M\colon M \to M$,



(2.6)

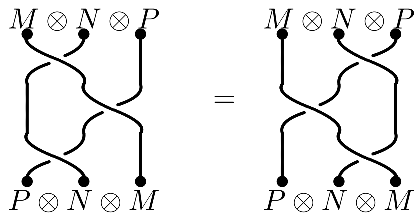
$$\check{R}_{MN}(\mathrm{id}_N \otimes \tau_M) = (\tau_M \otimes \mathrm{id}_N)\check{R}_{MN},$$

and the relations in (???) imply that if M, N and P are U-modules then

(2.7)

$$\check{R}_{M,N\otimes P} = (\check{R}_{MN} \otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP}) \qquad \check{R}_{M\otimes N,P} = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \mathrm{id}_N),$$

as operators on $M \otimes N \otimes P$. The relations (2.9) and (2.10) together imply the braid relation



$$(\check{R}_{MN} \otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP})(\check{R}_{NP} \otimes \mathrm{id}_M) = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \mathrm{id}_N)(\mathrm{id}_P \otimes \check{R}_{MN}), \tag{2.8}$$

3 The quantum double $D(A)$

In general it can be very difficult to find quasitriangular Hopf algebras, especially ones where the element $\mathcal{R}$ is different from $1 \otimes 1$. The construction in (???) below says that, given a Hopf algebra A we can sort of paste it and its dual A^* together to get a quasitriangular Hopf algebra $D(A)$ and that the $\mathcal{R}$ for this new quasitriangular Hopf algebra is both a natural one and is nontrivial.

Let $A = (A, m, \Delta, \iota, \varepsilon, S)$ be a Hopf algebra over $\mathbb{K}$. Let $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ be the dual of A. There is a natural bilinear pairing $\langle , \rangle \colon A^* \otimes A \longrightarrow \mathbb{K}$ between A and A^* given by

$$\langle \alpha, a \rangle = \alpha(a), \qquad \text{for all } \alpha \in A^* \text{ and } a \in A.$$

Extend this notation so that if $\alpha_1, \alpha_2 \in A^*$ and $a_1, a_2 \in A$ then

$$\langle \alpha_1 \otimes \alpha_2, a_1 \otimes a_2 \rangle = \langle \alpha_1, a_1 \rangle \langle \alpha_2, a_2 \rangle.$$

We make A^* into a Hopf algebra, which is denoted $A^{*\mathrm{coop}}$, by defining a multiplication and a comultiplication Δ on A^* via the equations

$$\langle \alpha_1 \alpha_2, a \rangle = \langle \alpha_1 \otimes \alpha_2, \Delta(a) \rangle \quad \text{and} \quad \langle \Delta^{\mathrm{op}}(\alpha), a_1 \otimes a_2 \rangle = \langle \alpha, a_1 a_2 \rangle,$$

for all $\alpha, \alpha_1, \alpha_2 \in A^*$ and $a, a_1, a_2 \in A$. The definition of Δ^{op} is in (4.1).

(a) The identity in $A^{*\mathrm{coop}}$ is the counit $\varepsilon \colon A \to \mathbb{K}$.

(b) The counit of $A^{*\mathrm{coop}}$ is the map

$$\begin{array}{cccc} \varepsilon \colon & A^* & \longrightarrow & \mathbb{K} \\ & \alpha & \longmapsto & \alpha(1) \end{array} \qquad \text{where 1 is the identity in } A.$$

(c) The antipode of $A^{*\mathrm{coop}}$ is given by the identity $\langle S(\alpha), a \rangle = \langle \alpha, S^{-1}(a) \rangle$, for all $\alpha \in A^*$ and all $a \in A$.

We want to paste the algebras A and $A^{*\mathrm{coop}}$ together in order to make a quasitriangular Hopf algebra $D(A)$. There are three main steps

(1) We paste A and $A^{*\mathrm{coop}}$ together by letting

$$D(A) = A \otimes A^{*\mathrm{coop}}. \tag{3.1}$$

Write elements of $D(A)$ as $a\alpha$ instead of as $a \otimes \alpha$.

(2) We want the multiplication in $D(A)$ to reflect the multiplication in A and the multiplication in $A^{*\mathrm{coop}}$. Similarly for the comultiplication.

(3) We want the $\mathcal{R}$-matrix to be

$$\mathcal{R} = \sum_i b_i \otimes b^i,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is the dual basis in A^*.

The condition in (2) determines the comultiplication in $D(A)$,

$$\Delta(a\alpha) = \Delta(a)\Delta(\alpha) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)},$$

where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)}$. The condition in (2) doesn't quite determine the multiplication in $D(A)$. We need to be able to expand products like $(a_1\alpha_1)(a_2\alpha_2)$. If we knew

$$\alpha_1 a_2 = \sum_j b_j\beta_j, \qquad \text{for some elements } \beta_j \in A^{*\mathrm{coop}} \text{ and } b_j \in A,$$

then we would have

$$(a_1\alpha_1)(a_2\alpha_2) = \sum_j (a_1 b_j)(\beta_j\alpha_2)$$

which is a well defined element of $D(A)$. Miraculously, the condition in (3) and the equation

$$\mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a), \qquad \text{for all } a \in A,$$

force that if $\alpha \in A^{*\mathrm{coop}}$ and $a \in A$ then, in $D(A)$,

$$\alpha a = \sum_{\alpha,a} \langle \alpha_{(1)}, S^{-1}(a_{(1)})\rangle \langle \alpha_{(3)}, a_{(3)}\rangle a_{(2)}\alpha_{(2)}, \qquad \text{and}$$

$$a\alpha = \sum_{\alpha,a} \langle \alpha_{(1)}, a_{(1)}\rangle \langle \alpha_{(3)}, S^{-1}(a_{(3)})\rangle \alpha_{(2)}a_{(2)},$$

where, if Δ is the comultiplication in $D(A)$,

$$(\Delta\otimes \mathrm{id})\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \quad \text{and} \quad (\Delta\otimes \mathrm{id})\Delta(\alpha) = \sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}.$$

These relations completely determine the multiplication in $D(A)$. This construction is summarised in the following theorem.

Theorem 3.1. *Let A be a (finite dimensional) Hopf algebra over $\mathbb{K}$ and let $A^{*\mathrm{coop}}$ be the Hopf algebra $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ except with opposite comultiplication. Then there exists a unique quasitriangular Hopf algebra $(D(A), \mathcal{R})$ given by*

(1) The $\mathbb{K}$-linear map

$$\begin{array}{ccc} A \otimes A^* & \longrightarrow & D(A) \\ a \otimes \alpha & \longmapsto & a\alpha \end{array}$$

is bijective.

(2) $D(A)$ contains A and $A^{*\mathrm{coop}}$ as Hopf subalgebras.

(3) The element $\mathcal{R} \in D(A) \otimes D(A)$ is given by

$$\mathcal{R} = \sum_i b_i \otimes b^i,$$

where $\{b_i\}$ is a basis of A and $\{b^i\}$ is the dual basis in $A^{*\mathrm{coop}}$.

In condition (2) of the theorem, A is identified with the image of $A \otimes 1$ under the map in (1) and $A^{*\mathrm{coop}}$ is identified with the image of $1 \otimes A^{*\mathrm{coop}}$ under the map in (1).

The following proposition constucts an ad-invariant bilinear form on $D(A)$.

Proposition 3.2. *Let A be a Hopf algebra. The bilinear form on the quantum double $D(A)$ of A which is defined by*

$$\langle a\alpha, b\beta\rangle = \langle \beta, S(a)\rangle\langle \alpha, S^{-1}(b)\rangle, \qquad \text{for all } a, b \in A \text{ and all } \alpha, \beta \in A^{*\mathrm{coop}},$$

satisfies

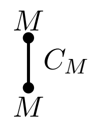
$$\langle \mathrm{ad}_a(x), y\rangle = \langle x, \mathrm{ad}_{S(u)}(y)\rangle \qquad \text{and} \qquad \langle y, x\rangle = \langle x, S^2(y)\rangle,$$

for all $u, x, y \in D(A)$.

4 The Casimir

There is also a *quantum Casimir element* $e^{-h\rho}u$ in the center of U and, for a U module M we define

$$\begin{array}{rccc} C_M\colon & M & \longrightarrow & M \\ & m & \longmapsto & (e^{-h\rho}u)m \end{array}$$



The elements $\mathcal{R}$ and $e^{-h\rho}u$ satisfy relations (see [LR, (2.1-2.12)]) which imply that,

$$C_{M\otimes N} = (\check{R}_{MN}\check{R}_{NM})^{-1}(C_M \otimes C_N). \tag{4.1}$$

If M is a U module generated by a highest weight vector v^+ of weight λ then, by [Dr, Prop. 3.2],

$$C_M = q^{-\langle\lambda,\lambda+2\rho\rangle}\mathrm{id}_M. \tag{4.2}$$

Note that $\langle\lambda, \lambda+2\rho\rangle = \langle\lambda+\rho, \lambda+\rho\rangle - \langle\rho,\rho\rangle$ are the eigenvalues of the classical Casimir operator [Dx, 7.8.5]. If M is a finite dimensional $U_h\mathfrak{g}$ module then M is a direct sum of the irreducible modules $L(\lambda)$, $\lambda \in P^+$, and

$$C_M = \bigoplus_{\lambda\in P^+} q^{-\langle\lambda,\lambda+2\rho\rangle}P_\lambda,$$

where $P_\lambda\colon M \to M$ is the projection onto $M^{[\lambda]}$ in M. From the relation (???) it follows that if $M = L(\mu)$, $N = L(\nu)$ are finite dimensional irreducible $U_h\mathfrak{g}$ modules then $\check{R}_{MN}\check{R}_{NM}$ acts on the λ isotypic component $L(\lambda)^{\oplus c_{\mu\nu}^\lambda}$ of the decomposition

$$L(\mu)\otimes L(\nu) = \bigoplus_\lambda L(\lambda)^{\oplus c_{\mu\nu}^\lambda} \qquad \text{by the constant} \qquad q^{\langle\lambda,\lambda+2\rho\rangle-\langle\mu,\mu+2\rho\rangle-\langle\nu,\nu+2\rho\rangle}. \tag{4.3}$$

5 Inner products

Let A be a Hopf algebra with antipode S and let M be an A-module. A bilinear form

$$\begin{array}{cccc} \langle,\rangle\colon & M \otimes M & \longrightarrow & \mathbb{K} \\ m\otimes n & \longmapsto & \langle m,n\rangle \end{array} \qquad \text{is } \textit{invariant} \text{ if} \qquad \langle am_1, m_2\rangle = \langle m_1, S(a)m_2\rangle,$$

for all $a \in A$, $m_1, m_2 \in M$. This is equivalent to the condition that the map $\langle,\rangle$ is a homomorphism of A-module when we identify $\mathbb{K}$ with the trivial A-module $\mathbf{1}$.

A bilinear form

$$\langle,\rangle\colon A\otimes A \to \mathbb{K} \qquad \text{is } \textit{ad-invariant} \text{ if} \qquad \langle \mathrm{ad}_a(b_1), b_2\rangle = \langle b_1, \mathrm{ad}_{S(a)}(b_2)\rangle,$$

for all $b_1, b_2. \in A$. In other words, the bilinear form is invariant if we view A as an A-module via the adjoint action.

6 Examples of Hopf algebras

7 Spectral subalgebras

Then

$$C_0 = \{\mu \in A^* \mid \mu(xy) = \mu(yx)\} \quad \text{is a commutative algebra,}$$

since, if $\ell_1, \ell_2 \in C_0$ and $a \in A$ then

$$\begin{aligned}(\ell_2\ell_1)(a) &= (\ell_1\otimes\ell_2)\Delta^{\mathrm{op}}(a) = (\ell_1\otimes\ell_2)\mathcal{R}\Delta(a)\mathcal{R}^{-1} \\ &= (\ell_1\otimes\ell_2)\Delta(a)\mathcal{R}^{-1}\mathcal{R} = (\ell_1\otimes\ell_2)\Delta(a) = (\ell_1\ell_2)(a),\end{aligned}$$

where the third equality uses the definition of C_0.

If $(A,\mathcal{R})$ is a quasitriangular Hopf algebra then $\mathcal{R}$ satisfies the *quantum Yang-Baxter equation* (QYBE),

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{12}(\Delta\otimes\mathrm{id})(\mathcal{R}) = (\Delta^{\mathrm{op}}\otimes\mathrm{id})(\mathcal{R})\mathcal{R}^{12} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}. \tag{7.1}$$

Since

$$\begin{aligned}\mathcal{R} &= (\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\mathcal{R}) = (\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})\mathcal{R}^{13}\mathcal{R}^{23} = (\varepsilon\otimes\mathrm{id})(\mathcal{R})\cdot\mathcal{R}, \quad \text{and} \\ \mathcal{R} &= (\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)(\mathrm{id}\otimes\Delta)(\mathcal{R}) = (\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)\mathcal{R}^{13}\mathcal{R}^{23} = (\mathrm{id}\otimes\varepsilon)(\mathcal{R})\cdot\mathcal{R},\end{aligned}$$

and so

$$(\varepsilon\otimes\mathrm{id})(\mathcal{R}) = 1 \qquad \text{and} \qquad (\mathrm{id}\otimes\varepsilon)(\mathcal{R}) = 1. \tag{7.2}$$

Then, since

$$\mathcal{R}(S\otimes\mathrm{id})(\mathcal{R}) = (m\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathcal{R}^{13}\mathcal{R}^{23}) = (m\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\mathcal{R}) = (\varepsilon\otimes\mathrm{id})(\mathcal{R}) = 1,$$

it follows that

$$(S\otimes\mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}. \tag{7.3}$$

Applying this to the pair $(A^{\mathrm{op}}, \mathcal{R}^{21})$ gives $(S^{-1}\otimes\mathrm{id})(\mathcal{R}^{21}) = (\mathcal{R}^{21})^{\mathrm{op}}$, and so

$$(\mathrm{id}\otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}. \tag{7.4}$$

Then

$$(S\otimes S)(\mathcal{R}) = (\mathrm{id}\otimes S)(S\otimes\mathrm{id})(\mathcal{R}) = (\mathrm{id}\otimes S)(\mathcal{R}^{-1}) = (\mathrm{id}\otimes S)(\mathrm{id}\otimes S^{-1}(\mathcal{R}) = \mathcal{R}. \tag{7.5}$$

The map $\phi\colon C \to Z(A)$ in the following proposition is an analogue of the Harish-Chandra homomorphism.

Proposition 7.1. *Let $(A, \mathcal{R})$ be a quasitriangular Hopf algebra. Then*

$$C = \{\lambda \in A^* \mid \lambda(xy) = \lambda(yS^2(x))\} \quad \textit{is a commutative algebra}$$

and the map

$$\begin{array}{rccc} \phi\colon & C & \longrightarrow & Z(A) \\ & \ell & \longmapsto & (\mathrm{id} \otimes \ell)(\mathcal{R}_{21}\mathcal{R}) \end{array}$$

is a well defined algebra homomorphism.

Proof. If $\ell_1, \ell_2 \in A^*$ and $a \in A$ then

$$\begin{aligned} (\ell_2\ell_1)(a) &= (\ell_1 \otimes \ell_2)\Delta^{\mathrm{op}}(a) = (\ell_1 \otimes \ell_2)(\mathcal{R}\Delta(a)\mathcal{R}^{-1}) \\ &= (\ell_1 \otimes \ell_2)(\Delta(a)\mathcal{R}^{-1}(S^2 \otimes S^2)(\mathcal{R})), \quad \text{by definition of } C, \\ &= (\ell_1 \otimes \ell_2)(\Delta(a)\mathcal{R}^{-1}\mathcal{R}), \qquad \text{by (???)}, \\ &= (\ell_1 \otimes \ell_2)(\Delta(a)) \\ &= (\ell_1\ell_2)(a), \end{aligned}$$

and hence C is a commutative algebra.

Let $a \in A$. First note that

$$\begin{aligned} a \otimes 1 &= (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\mathrm{id} \otimes m)(\mathrm{id} \otimes S^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{\mathrm{op}})\Delta(a) \\ &= \sum_a a_{(1)} \otimes S^{-1}(a_{(3)})a_{(2)} = \sum_a (1 \otimes S^{-1}(a_{(2)}))(a_{(11)} \otimes a_{(12)}) \\ &= \sum_a (1 \otimes S^{-1}(a_{(2)}))\Delta(a), \end{aligned}$$

since S^{-1} is the antipode of A^{op}, and

$$\begin{aligned} a \otimes 1 &= (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \mathrm{id} \otimes S)(\mathrm{id} \otimes \Delta)\Delta(a) \\ &= \sum_a a_{(1)} \otimes a_{(2)}S(a_{(3)}) = \sum_a (a_{(11)} \otimes a_{(12)})(1 \otimes S(a_{(2)})) \\ &= \sum_a \Delta(a_{(1)})(1 \otimes S(a_{(2)})). \end{aligned}$$

Then, since

$$\mathcal{R}^{21}\mathcal{R}\Delta(a) = \mathcal{R}^{21}\Delta^{\mathrm{op}}(a)\mathcal{R} = \Delta(a)\mathcal{R}^{21}\mathcal{R},$$

$$\begin{aligned} a\phi(\ell) &= a(\mathrm{id} \otimes \ell)(\mathcal{R}^{21}\mathcal{R}) = (\mathrm{id} \otimes \ell)((a \otimes 1)\mathcal{R}^{21}\mathcal{R}^{12}) \\ &= (\mathrm{id} \otimes \ell)\left(\sum_a (1 \otimes S^{-1}(a_{(2)}))\Delta(a_{(1)})\mathcal{R}^{21}\mathcal{R}\right) \\ &= (\mathrm{id} \otimes \ell)\left(\sum_a \Delta(a_{(1)})\mathcal{R}^{21}\mathcal{R}(1 \otimes S(a_{(2)})\right), \quad \text{by definition of } C, \\ &= (\mathrm{id} \otimes \ell)\left(\mathcal{R}^{21}\mathcal{R}\sum_a \Delta(a_{(1)})(1 \otimes S(a_{(2)}))\right) \\ &= (\mathrm{id} \otimes \ell)(\mathcal{R}^{21}\mathcal{R}(a \otimes 1) = (\mathrm{id} \otimes \ell)(\mathcal{R}^{21}\mathcal{R})a = \phi(\ell)a, \end{aligned}$$

and so $\phi(\ell) \in Z(A)$. Since

$$\begin{aligned}\phi(\ell_1\ell_2) &= (\mathrm{id}\otimes \ell_1\ell_2)(\mathcal{R}^{21}\mathcal{R}) = (\mathrm{id}\otimes \ell_1\otimes \ell_2)((\mathrm{id}\otimes\Delta)(\mathcal{R}^{21}\mathcal{R}))\\ &= (\mathrm{id}\otimes \ell_1\otimes \ell_2)(\mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12}) = (\mathrm{id}\otimes \ell_1)(\mathcal{R}^{21}(\phi(\ell_2)\otimes 1)\mathcal{R}^{12})\\ &= (\mathrm{id}\otimes \ell_1)(\mathcal{R}^{21}\mathcal{R}(\phi(\ell_2)\otimes 1)), \quad \text{since } \phi(\ell_2)\in Z(A),\\ &= \phi(\ell_1)\phi(\ell_2),\end{aligned}$$

and so ϕ is a homomorphism. $\square$

8 RTT realizations

Let A be a Hopf algebra with an invertible element

$$\mathcal{R} = \sum_r a_r \otimes b_r \quad \in A\otimes A \qquad \text{such that} \qquad \mathcal{R}\Delta(a)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(a),$$

for $a \in A$. The dual A^* of A is a Hopf algebra. Fix a positive integer n and an index set $\hat{T}$. Let

$$\{\rho^\lambda \colon A \to M_n(\mathbb{C}) \mid \lambda \in \hat{T}\}$$

be a set of representations of A. Their matrix entries

$$\rho^\lambda_{ij}\colon A \to \mathbb{C} \quad \text{are elements of } A^*.$$

On the ρ^λ_{ij}, the coproduct $\Delta\colon A^* \to A^*\otimes A^*$ has values

$$\Delta(\rho^\lambda_{ij}) = \sum_{k=1}^n \rho^\lambda_{ik}\otimes\rho^\lambda_{kj}, \qquad \text{since} \qquad \rho^\lambda_{ij}(u_1u_2) = \sum_{k=1}^n \rho^\lambda_{ik}(u_1)\rho^\lambda_{kj}(u_2),$$

for $u_1, u_2 \in A$. Let

$$\mathcal{R}(\lambda,\mu) = (\rho^\lambda\otimes\rho^\mu)(\mathcal{R}) \qquad \text{and} \qquad T(\lambda) = (\rho^\lambda_{ij}),$$

so that $T(\lambda)$ is a matrix in $M_n(A^*)$. Then

$$T(\lambda)\otimes \mathrm{id} = \textstyle\sum_{i,j,k} t^\lambda_{ij}(E_{ij}\otimes E_{kk}), \qquad \mathrm{id}\otimes T(\mu) = \sum_{i,j,k} t^\mu_{k\ell}(E_{ii}\otimes E_{k\ell}), \qquad \text{and}$$

$$\mathcal{R}(\lambda,\mu) = \sum_{i,j,k,\ell} \rho^\lambda_{ij}(a_r)\rho^\mu_{k\ell}(b_r)(E_{ij}\otimes E_{k\ell}).$$

Since

$$\mathcal{R}(\lambda,\mu)(T(\lambda)\otimes\mathrm{id})(\mathrm{id}\otimes T(\mu)) = \sum_{\substack{i,j,k,\ell\\ x,y}} \rho^\lambda_{ix}(a_r)t^\lambda_{xj}\rho^\mu_{xy}(b_r)t^\mu_{y\ell}\ (E_{ij}\otimes E_{k\ell}), \qquad \text{and}$$

$$(\mathrm{id}\otimes T(\mu))(T(\lambda)\otimes\mathrm{id})\mathcal{R}(\lambda,\mu) = \sum_{\substack{i,j,k\ell\\ \alpha,\beta}} t^\mu_{k\beta}t^\lambda_{i\alpha}\rho^\lambda_{\alpha j}(a_s)\rho^\mu_{\beta\ell}(b_s),$$

the equation

$$\mathcal{R}(\lambda,\mu)(T(\lambda)\otimes\mathrm{id})(\mathrm{id}\otimes T(\mu)) = (\mathrm{id}\otimes T(\mu))(T(\lambda)\otimes\mathrm{id})\mathcal{R}(\lambda,\mu)$$

is a concise way of encoding the relations

$$\begin{aligned}
\left(\sum_{x,y} \rho^\lambda_{ix}(a_r)\rho^\mu_{ky}(b_r)\rho^\lambda_{xj}\rho^\mu_{y\ell}\right)(a) &= \sum_{x,y,a} \rho^\lambda_{ix}(a_r)\rho^\mu_{ky}(b_r)\rho^\lambda_{xj}(a_{(1)}\rho^\mu_{y\ell}(a_{(2)})\\
&= \sum_a \rho^\lambda_{ij}(a_r a_{(1)})\rho^\mu_{k\ell}(b_r a_{(2)})\\
= (\rho^\lambda_{ij}\otimes\rho^\mu_{k\ell})(\mathcal{R}\Delta(a)) &= (\rho^\lambda_{ij}\otimes\rho^\mu_{k\ell})(\Delta^{\mathrm{op}}(a)\mathcal{R})\\
&= \sum_a \rho^\lambda_{ij}(a_{(2)}a_s)\rho^\mu_{k\ell}(a_{(1)}b_s)\\
&= \sum_{\alpha,\beta} \rho^\lambda_{i\alpha}(a_{(2)})\rho^\lambda_{\alpha j}(a_s)\rho^\mu_{k\beta}(a_{(1)})\rho^\mu_{\beta\ell}(b_s)\\
&= \left(\sum_{\alpha,\beta} \rho^\mu_{k\beta}\rho^\lambda_{i\alpha}\rho^\lambda_{\alpha_j}(a_s)\rho^\mu_{\beta\ell}(b_s)\right)(a)
\end{aligned}$$

which are satisfied by the ρ^λ_{ij} in A^*.

Let B be the Hopf algebra given by

$$\text{generators} \qquad t^\lambda_{ij}, \qquad 1\le i,j\le n, \quad \lambda\in\hat{T},$$

and relations

$$\mathcal{R}(\lambda,\mu)(T(\lambda)\otimes \mathrm{id})(\mathrm{id}\otimes T(\mu)) = (\mathrm{id}\otimes T(\mu))(T(\lambda)\otimes\mathrm{id})\mathcal{R}(\lambda,\mu)$$

with comultiplication given by

$$\Delta(t^\lambda_{ij}) = \sum_{k=1}^n t^\lambda_{ik}\otimes t^\lambda_{kj}.$$

The the map

$$\begin{array}{ccc} B & \longrightarrow & A^* \\ t^\lambda_{ij} & \longmapsto & \rho^\lambda_{ij} \end{array}$$

is a Hopf algebra homomorphism.

We really want a map $B\to A$, not $B\to A^*$. But it is "easy" to make maps $A^*\to A$. For example, one can construct a map $A^*\to A$ by

$$l\to(id\otimes l)(R) \quad \text{or} \quad l\to(id\otimes l)(R_{21}^{-1}) \quad \text{or} \quad l\to(id\otimes l)(R_{21}R).$$

In the case of Yangian or $U_q(\mathfrak{g})$, the composition $\Phi: B\to A^*\to A$ is surjective and $\ker\Phi$ is generated by the elments of the center of B.

References

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