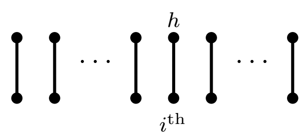
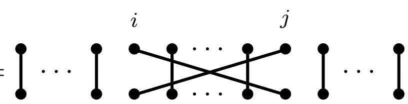
The wreath products $G_{H,1,k} = H \wr S_k$

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

1 The groups $G_{H,1,n}$

The semidirect product $G_{H,1,n} = H^n \rtimes S_n$ is the group of permutations with edges colored by elements of h. The product is the usual product on permutations with the convention that elements of H slide along edges and multiply when they collide. The group $G_{H,1,n}$ is generated by the elements

$t_i(h) =$ , for $1 \le i \le n$, $h \in H$,

$s_{ij} =$ , for $1 \le i < j \le n$,

and the subgroup H^n consists of the elements

$$t_h = t_1(h_1)t_2(h_2)\cdots t_n(h_n), \qquad \text{where } h = (h_1, \ldots, h_n),\ h_i \in H.$$

The operation in the semidirect product in the group

$$G_{H,1,n} = H^n \rtimes S_n = \{t_h w \mid h \in H^n, w \in S_n\}$$

is determined by the product in S_n, and

$$\begin{array}{ll} t_h t_k = t_{hk}, & \text{for } h, k \in H^n, \quad \text{and} \\ w t_h = t_{wh} w, & \text{where} \quad w(h_1, \ldots, h_n) = (h_{w(1)}, \ldots, h_{w(n)}), \end{array} \tag{1.1}$$

for $w \in S_n$ and $h = (h_1, \ldots, h_n) \in H^n$.

Let H^* be an index set for the conjugacy classes of H and let H_α, $\alpha \in H^*$, be a set of conjugacy class representatives. Let

$$G^*_{H,1,n} = \{\mu = (\mu^{(\alpha)})_{\alpha \in H^*} \mid \mu \text{ has } n \text{ boxes total}\},$$

be the set of H^*-multipartitions, tuples of partitions with components indexed by the elements of H^*, such that the total number of boxes in the multipartition is n. Then the elements

$$\gamma_\mu = PICTURE, \qquad \text{for } \mu \in G^*_{H,1,n},$$

are a set of conjugacy class representatives for $G_{H,1,n}$. The centralizer of γ_μ in $G_{H,1,n}$ is

$$Z_G(\gamma_\mu) = ???$$

with

$$\mathrm{Card}(Z_G(\mu)) = ???.$$

Each element of $G(r,p,n)$ is conjugate by elements of S_n to a disjoint product of cycles of the form

$$\xi_i^{\lambda_i} \cdots \xi_k^{\lambda_k}(i, i+1, \ldots, k).$$

By conjugating this cycle by $\xi_i^{-c}\xi_{i+1}^{\lambda_i}\xi_{i+2}^{\lambda_i+\lambda_{i+1}} \cdots \xi_k^{\lambda_i+\cdots+\lambda_{k-1}} \in G(r,r,n)$, we have

$$\xi_i^{-c}\xi_k^{c+\lambda_i+\cdots+\lambda_k}(i, \ldots, k), \qquad \text{where } \ c = (k-i)\lambda_i + (k-i-1)\lambda_{i+1} + \cdots + \lambda_{k-1}.$$

If $i_1, i_2, \ldots, i_\ell$ denote the minimal indices of the cycles and $c_1, \ldots, c_\ell$ are the numbers c for the various cycles, then after conjugating by $\xi_{i_1}^{c_1} \cdots \xi_{i_{\ell-1}}^{c_{\ell-1}} \xi_{i_\ell}^{-(c_1+\cdots+c_{\ell-1})} \in G(r,r,n)$, each cycle becomes

$$\xi_k^{\lambda_i+\cdots+\lambda_k}(i, \ldots, k) \qquad \text{except the last, which is} \qquad \xi_{i_\ell}^{-a}\xi_n^{b}(i_\ell, \ldots, n),$$

where $a = c_1 + \cdots + c_\ell$ and $b = a + \lambda_{i_\ell} + \cdots + \lambda_n$. If $k = n - i_\ell + 1$ is the length of the last cycle, then conjugating the last cycle by $\xi_{i_\ell}^{k-1}\xi_{i_\ell+1}^{-1} \cdots \xi_n^{-1} \in G(r,r,n)$ gives

$$\xi_{i_\ell}^{-a+k}\xi_n^{b-k}(i_\ell, \ldots, n).$$

If we conjugate the last cycle by $\xi_{i_\ell}^{p} \in G(r,p,n)$, we have

$$\xi_{i_\ell}^{-a+p}\xi_n^{b-p}(i_\ell, \ldots, n).$$

In summary, any element g of $G(r,p,n)$ is conjugate to a product of disjoint cycles where each cycle is of the form

$$\xi_k^{a}(i, i+1, \ldots, k), \qquad 0 \le a \le r-1,$$

except possibly the last cycle, which is of the form

$$\xi_{i_\ell}^{a}\xi_n^{b}(i_\ell, i_\ell+1, \ldots, n), \qquad \text{with } 0 \le a \le \gcd(p,k) - 1,$$

where $k = n - i_\ell + 1$ is the length of the last cycle.

Let $Z_{G(r,p,n)}(g) = \{h \in G(r,p,n) \mid hg = gh\}$ denote the centralizer of $g \in G(r,p,n)$. Since $G(r,p,n)$ is a subgroup of $G(r,1,n)$,

$$Z_{G(r,p,n)}(g) = Z_{G(r,1,n)}(g) \cap G(r,p,n),$$

for any element $g \in G(r,p,n)$. Suppose that g is an element of $G(r,1,n)$ which is a product of disjoint cycles of the form $\xi_k^a(i, \ldots, k)$ and that $h \in G(r,1,n)$ commutes with g. Conjugating g by h effects some combination of the following operations on the cycles of g:

(a) permuting cycles of the same type, $\xi_k^a(i, \ldots, k)$ and $\xi_m^b(j, \ldots, m)$ with $b = a$ and $k - i = m - j$,

(b) conjugating a single cycle $\xi_k^a(i, \ldots, k)$ by powers of itself, and

(c) conjugating a single cycle $\xi_k^a(i, \ldots, k)$ by $\xi_i^b \cdots \xi_k^b$, for any $0 \le b \le r-1$.

Furthermore, the elements of $G(r,1,n)$ which commute with g are determined by how they "rearrange" the cycles of g and a count (see [Mac, p. 170]) of the number of such operations shows that if $g \in G(r,1,n)$ and $m_{a,k}$ is the number of cycles of type $\xi^a_{i+k}(i, i+1, \ldots, i+k)$ for g, then

$$\mathrm{Card}(Z_{G(r,1,n)}(g)) = \prod_{a,k} (m_{a,k}! \cdot k^{m_{a,k}} \cdot r)\,. \tag{1.2}$$

Let $\hat{H}$ be an index set for the irreducible H modules. If $\gamma \in \hat{H}$ then let $\hat{H}^\gamma$ be an index set for a basis of H^γ so that

$$H^\gamma \quad \text{has basis} \quad \{m_P \mid P \in \hat{H}^\gamma\}, \qquad \text{with } H\text{-action} \qquad h\, m_P = \sum_{Q \in \hat{H}^\gamma} h_{QP} m_Q,$$

for appropriate constants $h_{QP} \in \mathbb{C}$.

Let $\lambda = (\lambda^{(\alpha)})_{\alpha \in \hat{H}}$ be a $\hat{H}$-tuple of partitions with n boxes total. A *standard tableau* of shape λ is a filling of the boxes of λ with $1, 2, \ldots, n$ such that, in each partition $\lambda^{(\alpha)}$,

(a) the rows increase from left to right,

(b) the columns increase from top to bottom.

The rows and columns of each partition $\lambda^{(\alpha)}$ are numbered as for matrices and

$$\begin{aligned} T(i) &\ \text{ is the box containing } i \text{ in } T, \\ c(b) &= j - i, \ \text{ if } b \text{ is in position } (i,j), \text{ and} \\ s(b) &= \alpha, \ \text{ if } b \text{ is in } \lambda^{(\alpha)}, \end{aligned}$$

The numbers $c(b)$ and $s(b)$ are the *content* and the $\hat{H}$*-type* of the box b, respectively.

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Theorem 1.1. *Use notations as in (???) and (???).*

(a) The irreducible representations G^λ_n of the group $G_{H,1,n} = H^n \ltimes S_n$ are indexed by the set

$$\hat{G}_n = \{\lambda = (\lambda^{(\alpha)})_{\alpha \in \hat{H}} \mid \lambda \textit{ has } n\textit{-boxes total}\},$$

of $\hat{H}$ multipartitions with n boxes total.

(b) $\dim G^\lambda_n = \sum_{\alpha \in \hat{H}} \dim(H^\alpha) \dim(S_n^{\lambda^{(\alpha)}})$.

(c) The irreducible $G_{H,1,n}$ module

$$G^\lambda_{r,1,n} \qquad \textit{has basis} \qquad \{(m_{P^{(1)}} \otimes \cdots \otimes m_{P^{(n)}}) \otimes v_T \mid T \in \hat{S}^\lambda, P^{(i)} \in \hat{H}^{s(T(i))}\}$$

with $G_{H,1,n}$ action given by

$$\begin{aligned} t_i(h)(m_{P^{(1)}} \otimes \cdots \otimes m_{P^{(n)}} \otimes v_T) &= m_{P^{(1)}} \otimes \cdots \otimes h m_{P^{(i)}} \otimes \cdots \otimes m_{P^{(n)}} \otimes v_T, \\ s_i(m_{P^{(1)}} \otimes \cdots \otimes m_{P^{(n)}} \otimes v_T) &= s_i(m_{P^{(1)}} \otimes \cdots \otimes m_{P^{(n)}}) \otimes s_i v_T, \end{aligned}$$

where

$$s_i v_T = (s_i)_{TT} v_T + (1 + (s_i)_{TT})\, v_{s_i T},$$

with

$$(s_i)_{TT} = \begin{cases} \dfrac{1}{c(T(i)) - c(T(i-1))}, & \text{if } s(T(i)) = s(T(i-1)), \\ 0, & \text{if } s(T(i)) \neq s(T(i-1)), \end{cases}$$

$c(T(i))$ is the content of the box containing i in T,

s_iT is the same as T except that i and $i-1$ are switched,

$v_{s_iT} = 0$ if s_iT is not standard.

Proof. The following argument determining the simple $G_{H,1,n}$ modules is often called *Clifford theory*. Let G^λ be a simple $G_{H,1,n} = H^n \rtimes S_n$ module. Let H^γ be a simple H^n submodule of G^λ. Then wH^γ is another simple H^n submodule of G^λ and

$$G^\lambda = \sum_{w \in S_n} wH^\gamma,$$

since the right hand side is an $H^n \rtimes S_n$ submodule of G^λ. Let

$$S_\gamma = \{w \in S_n \mid wH^\gamma \cong H^\gamma\} = \{w \in S_n \mid w\gamma = \gamma\}.$$

Thus

$$G^\lambda = \sum_{w_i \in S_n/S_\gamma} w_i N = \mathrm{Ind}_{H^n \rtimes S_\gamma}^{H^n \rtimes H_n}(N), \qquad \text{where} \qquad N = \sum_{w \in S_\gamma} wH^\gamma.$$

Then

$$N \cong H^\gamma \otimes S_\gamma^\lambda \qquad \text{with action} \qquad hw(m \otimes v) = hwm \otimes wm,$$

where S_γ^λ is a simple S_γ module. Since we are free to choose γ in its S_n orbit we may assume that γ is of the form

$$\gamma = (\underbrace{\gamma_1, \ldots, \gamma_1}_{\mu_1 \text{ times}}, \underbrace{\gamma_2, \ldots, \gamma_2}_{\mu_2 \text{ times}} \cdots, \underbrace{\gamma_\ell, \ldots, \gamma_\ell}_{\mu_\ell \text{ times}}) \qquad \text{so that} \qquad S_\gamma = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_\ell},$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ is a partition of n. An irreducible representation of S_γ is indexed by a tuple of partitions, one partition for each γ_i that appears in γ, so that the total number of boxes in the tuple of partitions is n.

Let us make this construction more explicit. Using the notation in (???), the simple H^n modules are indexed by the set $\hat{H}^n$ and a simple H^n module

$$H^{(\gamma_1, \ldots, \gamma_n)} \qquad \text{has basis} \qquad \{m_{P^{(1)}} \otimes \cdots m_{P^{(n)}} \mid P^{(i)} \in \hat{H}^{\gamma_i}\}.$$

Then the action of S_n on H^n modules in (???) is given by

$$w(m_{P^{(1)}} \otimes \cdots m_{P^{(n)}}) = m_{P^{(w(1))}} \otimes \cdots m_{P^{(w(n))}}, \qquad \text{for } w \in S_n,$$

defines an action of S_n on the irreducible H^n modules. The resulting action of S_n on $\hat{H}^n$ is given by

$$w(\gamma_1, \ldots, \gamma_n) = (\gamma_{w(1)}, \ldots, \gamma_{w(n)}).$$

Returning to the setup in equation (???),

$$wH^\gamma = H^{w\gamma}, \qquad \text{for } w \in S_n.$$

(The fact that $wH^\gamma = H^{w\gamma}$ means that in this case the cocycles (factor sets) that appear in Clifford theory are trivial.) $\square$

The *Casimir element* is the sum of the elements in the conjugacy class of s_{12},

$$\kappa_n = \sum_{1\le i<j\le n} \sum_{h\in H} t_i(h)t_j(h^{-1})s_{ij},$$

with notations as in (???).

Theorem 1.2.

(a) The Casimir element κ_n for $G_{H,1,n}$ is a central element of the group algebra of $G_{H,1,n}$ such that

$$\kappa_n \text{ acts on } G_n^\lambda \text{ by the constant } \quad \sum_{b\in\lambda} c(b).$$

(b) Let H^* be an index set for the conjugacy classes of H and let $\mu \in H^*$. Let

$$z(\mu) = \sum_{h\in\mathcal{C}_\mu} \sum_{i=1}^{n} t_i(h),$$

where the sum is over all elements of H in the conjugacy class μ. Then $z(\mu)$ is an element of the center of the group algebra of $G_{H,1,n}$ and

$$z(\mu) \text{ acts on } G_n^\lambda \text{ by the constant } \quad \sum_{\alpha\in\hat{H}} \frac{\chi_H^\alpha(\mu)}{\dim(H^\alpha)}.$$

Proof.

$$\begin{aligned}\frac{1}{|H|}\sum_{h\in H} hm_Q \otimes h^{-1}m_P &= \sum_{h,R,S} A_{RQ}^\alpha(h)A_{SP}^\beta(h^{-1})(m_R\otimes m_S)\\ &= \sum_{R,S}(m_R\otimes m_S)\sum_{h\in H} A_{RQ}^\alpha(h)A_{SP}^\beta(h^{-1}).\end{aligned}$$

Define an element of $\mathrm{Hom}(H^\alpha, H^\beta)$,

$$\varphi_{QS}^{\alpha\beta} \colon H^\alpha \to H^\beta, \qquad \text{by} \qquad \varphi_{QS}^{\alpha\beta}(m_P) = \delta_{QP}m_S.$$

Then as elements of $\mathrm{Hom}(H^\alpha, H^\beta)$,

$$g\sum_{h\in H} h\varphi_{QS}^{\alpha\beta}h^{-1} = \sum_{h\in H} gh\varphi_{QS}^{\alpha\beta}(gh)^{-1}g,$$

and

$$\mathrm{Tr}\left(\sum_{h\in H} h\varphi_{QS}^{\alpha\alpha}h^{-1}\right) = \mathrm{Tr}\left(\sum_{h\in H} \varphi_{QS}^{\alpha\alpha}\right) = |H|\delta_{QS},$$

and thus, by Schur's lemma,

$$\sum_{h\in H} h\varphi_{QS}^{\alpha\beta}h^{-1} = \frac{|H|}{d_\lambda}\delta_{\alpha\beta}\delta_{QS}\cdot \mathrm{id}$$

where $d_\alpha = \dim(H^\alpha)$. Thus

$$\sum_{h\in H} A^\alpha_{RQ}(h) A^\beta_{SP}(h^{-1}). = \left(\sum_{h\in H} A^\alpha(h)\varphi_{QS} A^\beta(h^{-1})\right)_{RP} = \frac{|H|}{d_\alpha}\delta_{\alpha\beta}\delta_{QS}\delta_{RP}.$$

Hence

$$\frac{1}{|H|}\sum_{h\in H} hm_Q \otimes h^{-1}m_P = \frac{1}{d_\alpha}\delta_{\alpha\beta}(m_P\otimes m_Q)$$

(a) Let $x_1 = 0$ and, for $2 \le k \le n$ let

$$x_k = \sum_{h\in H}\sum_{1\le i<k} t_i(h)t_k(h^{-1})s_{ik} \qquad \text{so that} \qquad x_1 + x_2 + \cdots + x_n = \kappa_n. \tag{1.3}$$

Then

$$x_k = s_k x_k s_k + \sum_{h\in H} t_{k-1}(h) s_k t_{k-1}(h^{-1}). \tag{1.4}$$

and

$$\begin{aligned}
x_k(m_P\otimes v_T) &= \left(s_k x_{k-1} s_k + \sum_{h\in H} s_k t_k(h) t_{k-1}(h^{-1})\right)(m_P\otimes v_T) \\
&= s_k\Big(c(T(k-1))(s_k)_{TT}(s_k m_P\otimes v_T) + c(T(k))(1+(s_k)_{TT})(s_k m_P\otimes v_{s_kT}) \\
&\qquad + \sum_{h\in H} t_{k-1}(h^{-1})t_k(h)(m_P\otimes v_T)\Big) \\
&= s_k\Big(c(T(k))s_k(m_P\otimes v_T) + (c(T(k-1)) - c(T(k)))(s_k)_{TT}(m_P\otimes v_T) \\
&\qquad + \sum_{h\in H} t_{k-1}(h^{-1})t_k(h)(m_P\otimes v_T)\Big) \\
&= \begin{cases} s_k(c(T(k))s_k v_T + ((-1)+1)v_T, & \text{if } s(T(k)) = s(T(k-1)), \\ s_k(c(T(k))s_k v_T + (0+0)v_T, & \text{if } s(T(k)) \ne s(T(k-1)), \end{cases} \\
&= c(T(k))v_T.
\end{aligned}$$

(b) Since $\sum_{h\in\mathcal{C}_\mu}$ acts on H^α by the constant

$$\frac{\chi^\alpha_H(\mu)}{\dim(H^\alpha)}$$

it follows that

$$\left(\sum_{h\in\mathcal{C}_\mu} t_i(h)\right)(m_P\otimes v_T) = \sum_{h\in\mathcal{C}_\mu}(t_i(h)m_P\otimes v_T) = \frac{\chi_H^{(\alpha_i)}(\mu)}{\dim(H^{(\alpha_i)})}(m_P\otimes v_T).$$

Part (b) of the theorem now follows by summing over i. □

For example, if

$$\begin{array}{lll} H = \mathbb{Z}/r\mathbb{Z}, & \text{then} & \hat{H} = \{0,1,2,\ldots,r-1\}, \\ H = \mathbb{Z}, & \text{then} & \hat{H} = \mathbb{C}^*, \\ H = \mathbb{C}^*, & \text{then} & \hat{H} = \mathbb{Z}, \end{array}$$

where, in the last case $\hat{H}$ indexes the rational representations of $H = \mathbb{C}^*$.

1.1 Characters of $G_{H,1,n}$

Let $\mu \in G_n^*$. Then

$$\left|Z_{G_n}(\mu)\right| = \prod_{\beta\in H^*} \left(\left|Z_{S_n}(\mu^{(\beta)})\right| \cdot \left|Z_H(\beta)\right|^{\ell(\mu^{(\beta)})}\right).$$

Let H^* be an index set for the conjugacy classes of H and, for each $\beta \in H^*$, let

$$x^{(\beta)} = \{x_1^{(\beta)}, x_2^{(\beta)}, \ldots\} \qquad \text{be a set of variables indexed by } \beta,$$

and let

$$p_\mu(x) = \prod_{\beta\in H^*} p_{\mu^{(\beta)}}\big(x^{(\beta)}\big), \qquad \text{for } \mu \in G_n^*,$$

so that $p_\mu(x)$ is the product of power symmetric functions from each of the variable sets $x^{(\beta)}$. Define a "change of variables" from the $x^{(\beta)}$ variables, which are indexed by $\beta \in H^*$, to $y^{(\alpha)}$ variables indexed by $\alpha \in \hat{H}$, by setting

$$p_r(y^{(\alpha)}) = \sum_{\beta\in H^*} \frac{\chi_H^\alpha(\beta) p_r\big(x^{(\beta)}\big)}{\left|Z_H(\beta)\right|}, \qquad \text{for each } \alpha \in \hat{H}, \text{ and each } r \in \mathbb{Z}_{>0}.$$

Define

$$s_\lambda(y) = \prod_{\alpha\in\hat{H}} s_{\lambda^{(\alpha)}}\big(y^{(\alpha)}\big), \qquad \text{for } \lambda \in \hat{G}_n.$$

Then

$$s_\lambda(y) = \sum_{\mu\in\hat{G}_n^*} \frac{\chi_{G_n}^\lambda(\mu) p_\mu(x)}{\left|Z_{G_n}(\mu)\right|} \qquad \text{and} \qquad p_\mu(x) = \sum_{\lambda\in\hat{G}_n} \overline{\chi_{G_n}^\lambda(\mu)} s_\lambda(y),$$

for $\lambda \in \hat{G}_n$ and $\mu \in G_n^*$.

2 The groups $G_{H,H/K,n}$

Let H be a group. Assume that H is abelian so that there is a well defined map $\phi\colon G_{H,1,n} \to H$ given by $\phi(t_h w) = h_1 \cdots h_n$, for $h = (h_1, \ldots h_n) \in H^n$ and $w \in S_n$. Let K be a subgroup of H and define a normal subgroup $G_{H,H/K,n}$ of $G_{H,1,n}$ by the exact sequence

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & G_{H,H/K,n} & \longrightarrow & G_{H,1,n} & \longrightarrow & H/K & \longrightarrow & \{1\} \\ & & & & t_h w & \longmapsto & h_1 h_2 \cdots h_n. & & \end{array}$$

Thus

$$G_{H,H/K,n} = \{t_h w \mid h_1 h_2 \cdots h_n \in K\} \qquad \text{with order} \qquad |G^K| = |H|^{n-1}|K|n!.$$

Let H be an abelian group and let $\hat{H}$ be an index set for the simple H-modules. The *dual group* is the set $\hat{H}$ with the operation induced by tensor product of H-modules. If $H = \mathbb{C}^*$ then the irreducible representations of H (as an algebraic group) are

$$X^k\colon \begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \\ x & \longmapsto & x^k \end{array} \quad \text{for } k \in \mathbb{Z}, \qquad \text{and} \qquad X^k X^\ell = X^{k+\ell},$$

so that $\hat{H} \cong \mathbb{Z}$. If $H = (\mathbb{C}^*)^n$ then $\hat{H}$ is a lattice,

$$\hat{H} \cong \mathbb{Z}^n, \qquad \text{with} \qquad X^\lambda X^\mu = X^{\lambda+\mu},$$

for $\lambda, \mu \in \mathbb{Z}^n$. If H is a finite abelian group then

$$H \cong \mathbb{Z}/r_1\mathbb{Z} \oplus \mathbb{Z}/r_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_\ell\mathbb{Z} \qquad \text{and} \qquad \hat{H} \cong \mathbb{Z}/r_1\mathbb{Z} \oplus \mathbb{Z}/r_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_\ell\mathbb{Z},$$

a quotient of the lattice in (???), so that

$$\hat{H} = \{X^\lambda \mid \lambda = (\lambda_1, \ldots, \lambda_n) \ \text{ with } \ \lambda_i \in \mathbb{Z}/r_i\mathbb{Z}\}.$$

Let H be abelian and let $\hat{H}$ be the dual group of H. The group $\hat{H}$ acts on the group algebra of H^n by algebra automorphisms,

$$\begin{array}{rccc} X^\lambda\colon & \mathbb{C}H^n & \longrightarrow & \mathbb{C}H^n \\ & t_{(h_1,\ldots,h_n)} & \longmapsto & X^\lambda(h_1h_2\cdots h_n)t_{(h_1,\ldots,h_n)} \end{array}$$

and on the group algebra of $G_{H,1,n} = H^n \rtimes S_n$ by algebra automorphisms,

$$\begin{array}{rccc} X^\lambda\colon & \mathbb{C}G_{H,1,n} & \longrightarrow & \mathbb{C}G_{H,1,n} \\ & t_{(h_1,\ldots,h_n)}w & \longmapsto & X^\lambda(h_1h_2\cdots h_n)t_{(h_1,\ldots,h_n)}w \end{array}$$

Let $\hat{K}$ be a subgroup of $\hat{H}$. Then the subalgebra of $\mathbb{C}G_{H,1,n}$ fixed by $\hat{K}$ is

$$(\mathbb{C}G_{H,1,n})^{\hat{K}} = \text{span}\{t_{(h_1,\ldots,h_n)}w \mid X^\lambda(h_1h_2\cdots h_n) = 1, \text{ for all } X^\lambda \in \hat{K}\}.$$

Then

$$\begin{aligned}(\mathbb{C}G_{H,1,n})^{\hat{K}} &= \text{span}\{t_{(h_1,\ldots,h_n)}w \;\Big|\; h_1h_2\cdots h_n \in K\} \\ &= \mathbb{C}G_{H,H/K,n}, \qquad \text{where } K = \textstyle\bigcap_{\lambda\in\hat{K}} \ker(X^\lambda).\end{aligned}$$

3 The groups $G_{r,p,n}$

The group $G_{H,H/K,n}$ is

$$\text{denoted } G_{r,p,n} \quad \text{if } H \text{ is a cyclic group of order } r \quad \text{and} \quad H/K \text{ is order } p.$$

Note that p is not necessarily prime and p divides r. The group $G_{r,p,n}$ can be realised as the group of $n \times n$ matrices such that

(a) There is exactly one nonzero entry in each row and each column,

(b) The nonzero entries are rth roots of unity,

(c) The (r/p)th power of the product of the nonzero entries is 1.

Special cases of these groups are

(1) $G_{r,1,1}$, the cyclic group of order r,

(2) $G_{r,r,2}$, the dihedral group of order $2r$,

(3) $G_{1,1,n} = S_n$, the symmetric group, or Weyl group of type A,

(4) $G_{\infty,1,n}$ is the *affine symmetric group* or the *affine Weyl group of type A*,

(5) $G_{2,1,n} = WB_n$, the Weyl group of type B_n,

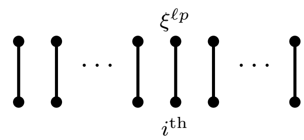
(6) $G_{2,2,n} = WD_n$, the Weyl group of type D_n,

All of these are subgroups of the group

$$N = (\mathbb{C}^*)^n \ltimes S_n \qquad \text{of monomial matrices in } GL_n(\mathbb{C})$$

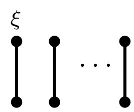
(the normalizer of the torus of diagonal matrices in $GL_n(\mathbb{C})$).

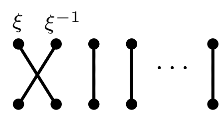
Let $\mathbb{Z}/r\mathbb{Z} = \{0, 1, \ldots, r-1\}$ and let $\xi = e^{2\pi i/r}$. The groups $G_{r,p,n}$ are complex reflection groups (generated by reflections). The *reflections* in $G_{r,p,n}$ are the elements

$$t_i^k t_j^{-k} s_{ij}, \qquad \text{and} \qquad t_i^{\ell p} = t_{(0,\ldots,0,\ell p,0,\ldots 0)} =$$ 

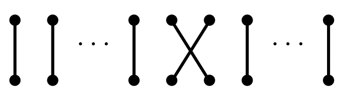
for $\leq i < j \leq n$, $0 \leq k \leq r-1$, and $0 \leq \ell \leq (r/p) - 1$.

Define

$$t_1 =$$ 

$$s_1 =$$ 

and

$$s_i = (i-1, i) =$$ 

for $2 \leq i \leq n$.

The group $G_{r,p,n}$ has a presentation by generators $t_1^p, s_1, s_2, \ldots, s_n$ and relations

$$
\begin{gathered}
(t_1^p)^{r/p} = 1 \qquad \text{and} \qquad s_i^2 = 1, \quad 1 \leq i \leq n, \\
s_i s_j = s_j s_i, \qquad \text{for } i, j \geq 2 \text{ such that } |i-j| > 1, \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad \text{for } 2 \leq i \leq n, \\
s_1 s_3 s_1 = s_3 s_1 s_3, \quad \text{and} \qquad s_1 s_j = s_j s_1, \quad \text{for } j > 3, \\
t_1^p s_2 t_1^p s_2 = s_2 t_1^p s_2 t_1^p, \quad \text{and} \qquad s_j t_1^p = t_1^p s_j, \quad \text{for } j > 2, \\
\underbrace{t_1^p s_1 t_1^p s_1 \cdots}_{2(r/p)\text{ factors}} = \underbrace{s_1 t_1^p s_1 t_1^p \cdots}_{2(r/p)\text{ factors}}, \qquad \text{and} \qquad \underbrace{s_1 s_2 s_1 \cdots}_{r \text{ factors}} = \underbrace{s_2 s_1 s_2 \cdots}_{r \text{ factors}},
\end{gathered}
$$

For $G_{r,1,n}$ the generator s_1 is unnecessary and, for $G_{r,r,n}$ the generator $t_1^r = 1$ and is irrelevant.

Note that only the groups ???? can be generated by n reflections.

4 Representations of the groups $G_{H,H/K,n}$

The irreducible representations of the group $G_{H,1,n}$ can be derived from Clifford theory or via the tower

$$\{1\} \subseteq G_{\frac{1}{2}} = G_1 \subseteq G_{\frac{3}{2}} \subseteq G_2 \subseteq G_{\frac{5}{2}} \subseteq G_3 \subseteq \cdots,$$

where

$$G_n = G_{H,1,n} = H^n \rtimes S_n \qquad \text{and} \qquad G_{n+\frac{1}{2}} = G_{H1,n} \times G_{H,1,1},$$

so that $G_{n+\frac{1}{2}}$ is a "Levi subgroup" of G_{n+1}. The tower $\hat{G}$ has

vertices on level n: multipartitions $\lambda = (\lambda^{(\alpha)})_{\alpha\in\hat{H}}$ with n boxes total,

vertices on level $n+\frac{1}{2}$: pairs $(\lambda, \Box_\alpha)$, where $\lambda \in \hat{G}_n$, $\alpha \in \hat{H}$.

edges from level n to level $n+\frac{1}{2}$:

$\lambda \to (\lambda, \Box_\alpha)$ for each $\alpha \in \hat{H}$.

edges from level $n+\frac{1}{2}$ to level n:

$(\mu, \Box_\alpha) \to \nu$ if ν is obtained from μ by adding a box to $\mu^{(\alpha)}$.

For each $0 \le m \le r-1$ the elements t_i^m, $1 \le i \le n$, form a conjugacy class in $G(r,1,n)$ and the elements $t_i^m t_j^{-m}(i,j)$, $1 \le i < j \le n$, $0 \le m \le r-1$, form another conjugacy class in $G(r,1,n)$. Thus the elements

$$z_s(m) = \sum_{i=1}^n t_i^m, \quad 0 \le m \le r-1, \qquad \text{and} \qquad z_\ell = \frac{1}{r}\sum_{m=0}^{r-1}\sum_{1\le i<j\le n} t_i^m t_j^{-m}(i,j),$$

are elements of $Z(\mathbb{C}G(r,1,n))$. So $z_s(m)$ and z_ℓ must act by a constant on any irreducible representation S^λ of $G(r,1,n)$. Define $x_1 = 0$,

$$x_k = \left(\sum_{\substack{1\le i<j\le k\\ 0\le \ell\le r-1}} t_i^\ell t_j^{-\ell}(i,j)\right) - \left(\sum_{\substack{1\le i<j\le k-1\\ 0\le \ell\le r-1}} t_i^\ell t_j^{-\ell}(i,j)\right) = \frac{1}{r}\sum_{1\le i<k,\ 0\le\ell\le r-1} t_i^\ell t_k^{-\ell}(i,k), \qquad \text{for } 2 \le k \le n, \text{ and}$$

$$y_k = \left(\sum_{i=1}^k t_i\right) - \left(\sum_{i=1}^{k-1} t_i\right) = t_k, \qquad \text{for } 1 \le k \le n.$$

Theorem 4.1. *The elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ all commute with each other and the action of these elements on the irreducible representation S^λ of $G(r,1,n)$ is given by*

$$y_k v_T = s(T(k)) v_T \qquad \text{and} \qquad x_k v_T = c(T(k)) v_T,$$

for all standard tableaux T.

Proof. The proof is by induction on k using the relations

$$x_k = s_k x_{k-1} s_k + \sum_{\ell=0}^{r-1} y_{k-1}^\ell s_k y_{k-1}^{-\ell} \qquad \text{and} \qquad y_k = s_k y_{k-1} s_k.$$

The base cases

$$x_1 v_T = 0 = c(T(1)) v_T \qquad \text{and} \qquad y_1 v_T = s(T(1)) v_T$$

are immediate from the definitions. Then

$$\begin{aligned}
y_k v_T &= s_k y_{k-1} s_k \\
&= s_k \left(s(T(k-1))(s_k)_{TT} v_T + (1 + (s_k)_{TT}) s(T(k)) v_{s_k T} \right) \\
&= s_k \left(s(T(k)) s_k v_T + (s(T(k-1)) - s(T(k)))(s_k)_{TT} v_T \right) \\
&= s(T(k)) v_T + 0 = s(T(k)) v_T, \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
x_k v_T &= \left(s_k x_{k-1} s_k + \frac{1}{r} \sum_{\ell=0}^{r-1} s_k y_{k-1}^{-\ell} y_k^{\ell} \right) v_T \\
&= s_k \left(c(T(k-1))(s_k)_{TT} v_T + c(T(k))(1 + (s_k)_{TT}) v_{s_k T} + \frac{1}{r} \sum_{\ell=0}^{r-1} s(T(k-1))^{-\ell} s(T(k))^{\ell} v_T \right) \\
&= s_k \left(c(T(k)) s_k v_T + (c(T(k-1)) - c(T(k)))(s_k)_{TT} v_T + \frac{1}{r} \sum_{\ell=0}^{r-1} \left(s(T(k))^{-1} s(T(k)) \right)^{\ell} v_T \right) \\
&= \begin{cases} s_k(c(T(k)) s_k v_T + ((-1) + 1) v_T, & \text{if } s(T(k)) = s(T(k-1)), \\ s_k(c(T(k)) s_k v_T + (0 + 0) v_T, & \text{if } s(T(k)) \neq s(T(k-1)), \end{cases} \\
&= c(T(k)) v_T.
\end{aligned}$$

$\square$

Define an action of H on $\hat{H}$ by

$$h\alpha = \chi_h \otimes \alpha,$$

and extend this to an action of H on the simple $G_{H,1,n}$ modules by

$$m_P \otimes v_T \longmapsto m_{hP} \otimes v_{hT}.$$

Theorem 4.2.

(a) The simple $G_{H,H/K,n}$ modules are indexed by pairs $(\bar{\lambda}, \mu)$ where $\bar{\lambda} \in K \backslash \hat{G}_n$, $\mu \in \hat{K}_\lambda$, where K_λ is the stabilizer of λ in K.

(b) The simple $G_{H,H/K,n}$ module, then for any fixed representative λ of the coset $\bar{\lambda}$,

$$G_{K,n}^{(\bar{\lambda},\mu)} = p_\mu G_n^{\lambda}, \qquad \text{where} \quad p_\mu = \sum_{k \in K_\lambda} \chi^\mu(k^{-1}) k,$$

is the minimal idempotent of K_λ corresponding to the module K_λ^μ.

(c) As a $(G_{H.H/K,n}, K_\lambda)$ bimodule

$$G^\lambda = \bigoplus_{\mu \in \hat{K}_\lambda} G_n^{(\lambda,\mu)} \otimes K_\lambda^\mu.$$

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