The 0 Hecke algebra

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1 The 0-Hecke algebra

Let W be a Weyl group with simple reflections $s_1, \ldots, s_n$. The 0-*Hecke algebra* is given by generators $T_1, \ldots, T_n$ and relations

$$T_i^2 = -T_i \qquad \text{and} \qquad \underbrace{T_iT_jT_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\text{ factors}}, \quad \text{for } i \neq j,$$

where m_{ij} is the order of s_is_j in W. If

$$e_i = -T_i, \qquad \text{and} \qquad f_i = 1 - e_i = 1 + T_i,$$

then

$$e_i^2 = e_i, \quad f_i^2 = f_i, \quad \underbrace{e_ie_je_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{e_je_ie_j\cdots}_{m_{ij}\text{ factors}}, \quad \underbrace{f_if_jf_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{f_jf_if_j\cdots}_{m_{ij}\text{ factors}}.$$

The last identity is proved by noting that a term of the form $\underbrace{1\cdot 1\cdots}_{k\text{ factors}}\cdot e_i \cdot 1 \cdot 1 \underbrace{\cdots}_{B}$ in the product $f_if_j\cdots = (1-e_i)(1-e_j)\cdots$ cancels with the term $1\cdots 1\cdot e_i \cdot 1 \cdot e_i \underbrace{\cdots}_{B}$. The remaining terms are products of the form

$$(-1)^{m_{ij}-k}\underbrace{1\cdots 1}_{k\text{ factors}}\underbrace{e_ie_je_i\cdots}_{m_{ij}-k\text{ factors}} \qquad \text{and} \qquad (-1)^{m_{ij}-k-1}\underbrace{1\cdots 1}_{k\text{ factors}}\underbrace{e_ie_je_i\cdots}_{m_{ij}-k-1\text{ factors}}\cdot 1.$$

Thus

$$\begin{aligned}
f_if_j\cdots &= 1 - e_i - e_j + e_ie_j + e_je_i - e_ie_je_i - e_je_ie_j + \cdots + \underbrace{e_ie_j\cdots}_{m_{ij}\text{ factors}} \\
&= 1 + \left(\sum_{k=1}^{m_{ij}-1} \underbrace{e_ie_j\cdots}_{k\text{ factors}}(-1)^k\right) + \underbrace{e_ie_j\cdots}_{m_{ij}\text{ factors}} \\
&= 1 + \left(\sum_{k=1}^{m_{ij}-1} \underbrace{e_ie_j\cdots}_{k\text{ factors}}(-1)^k\right) + \underbrace{e_je_i\cdots}_{m_{ij}\text{ factors}} = f_jf_i\cdots.
\end{aligned}$$

1.1 Irreducible representations

Let V be a simple $H(0)$ module and let $v \in V$, $v \neq 0$. Then

$$\mathbb{C}e_{w_0}v \qquad \text{is a submodule of } V.$$

So $V = \mathbb{C}e_{w_0}v$ or $e_{w_0}v = 0$. If $e_{w_0}v = 0$ then $\mathbb{C}e_{s_iw_0}v$ is a submodule of V, $1 \leq i \leq n$. So $V = \mathbb{C}e_{s_iw_0}v$ for some i, or all $e_{s_iw_0}v = 0$. Thus, by descending induction on $\ell(w)$ we find

$$V = \mathbb{C}e_w v, \quad \text{for some } w \in W.$$

So V is one dimensional.

Let $J \subseteq \{1, 2, \ldots, n\}$ and define

$$\chi^J : H(0) \to \mathbb{C} \qquad \text{by} \qquad \chi^J(e_i) = \begin{cases} 1, & \text{if } i \in J, \\ 0, & \text{if } i \notin J. \end{cases}$$

This defines 2^n irreducible one dimensional representations of $H(0)$. By the argument in the first paragraph, all irreducible representations of $H(0)$ are one dimensional. The relation $e_i^2 = e_i$ forces $\chi(e_i) = 0$ or $\chi(e_i) = 1$ for a one dimensional representation $\chi : H(0) \to \mathbb{C}$. so the

$$\chi^J, \qquad J \subseteq \{1, 2, \ldots, n\}$$

are a complete set of irreducible $H(0)$ representations.

1.2 The radical of $H(0)$

For each i, j, $1 \leq i, j \leq n$, $i \neq j$, and each $J \subseteq \{1, 2, \ldots, n\}$

$$\chi^J(e_ie_j - e_ie_je_i) = 0.$$

So the element $e_ie_j - e_ie_je_i$ acts by 0 on every irreducible $H(0)$-module. So

$$e_ie_j - e_ie_je_i \in \mathrm{Rad}(H(0)).$$

If

$$\overline{H} = \frac{H(0)}{\langle e_ie_j - e_ie_je_i \rangle}$$

and $\overline{e_i}$ is the image of e_i in $\overline{H}$ then

$$\begin{aligned} \overline{e_i}\overline{e_j} = \overline{e_i}\overline{e_j}\overline{e_i} = \overline{e_i}\overline{e_j}\overline{e_i}\overline{e_j} = \cdots &= \underbrace{\overline{e_i}\overline{e_j}\overline{e_i}\cdots}_{m_{ij}\text{ factors}} \\ &= \underbrace{\overline{e_j}\overline{e_i}\overline{e_j}\cdots}_{m_{ij}\text{ factors}} = \cdots = \overline{e_i}\overline{e_j}, \end{aligned}$$

for $i \neq j$. So $\overline{H}$ is a commutative algebra. In view of the relation $\overline{e_i}^2 = \overline{e_i}$ the elements

$$\overline{e_{i_1}} \cdots \overline{e_{i_k}}, \quad 1 \leq i_1 < \cdots < i_k \leq n, \qquad \text{span } \overline{H}$$

and so $\dim(\overline{H}) \leq 2^n$. Since all 2^n irreducible representations χ^J of $H(0)$ are representations of $\overline{H}$,

$$\dim(\overline{H}) = 2^n \qquad \text{and} \qquad \overline{H} \text{ is semisimple.}$$

So

$$\mathrm{Rad}(H(0)) = \langle e_ie_j - e_ie_je_j \mid \text{for } i \neq j \rangle.$$

1.3 Projective Indecomposable $H(0)$ modules

For $w \in W$ let $e_w = e_{i_1} \dots e_{i_p}$ and $f_w = f_{i_1} \cdots f_{i_p}$ for a reduced decomposition $w = s_{i_1} \cdots s_{i_p}$ of w.

Let $J \subseteq \{1, \dots, n\}$ and define

$$P(J) = H(0) e_{w_J} f_{w_{J^c}},$$

where w_J and w_{J^c} are the longest elements of the parabolic subgroups W_J and W_{J^c}, respectively. Then $P(J)$ has basis

$$\{e_w f_{w_{J^c}} \mid D_r(w) = J\}, \qquad \text{where} \quad D_r(w) = \{i \mid w s_i < w\}.$$

The *Cartan invariants* are

$$\begin{aligned} c_{JK} &= (\text{multiplicity of } L(K) \text{ in a composition series of } P(J)) \\ &= \mathrm{Card}\{w \in W \mid D_r(w) = J,\ D_\ell(w) = K\}, \end{aligned}$$

where $D_\ell(w) = \{i \mid s_i w < w\}$. Since

$$e_w f_{w_{J^c}} = e_w + \sum_{v > w} c_v e_v, \qquad \text{for some } c_v \in \mathbb{C},$$

it follows that

$$H(0) = \bigoplus_{J \subseteq \{1,2 \dots, n\}} P(J), \qquad \text{as } H(0)\text{-modules.}$$

Since $P(J)$ has head isomorphic to

$$L(J) \cong \frac{P(J)}{\mathrm{span}\{e_w e_{w_J} f_{w_{J^c}} \mid w > 1\}}$$

the $P(J)$ are the projective indecomposable modules (PIMs) for $H(0)$. The matrix

$$C = (c_{JK}) \qquad \text{is the Cartan matrix for } H(0).$$

1.4 Decomposition numbers

Let $L_q(\lambda)$ be the irreducible $H(q)$ modules in a form which can be specialized at $q = 0$. Let $L_0(K)$ denote the irreducible $H(0)$ module indexed by K. The *decomposition numbers* are

$$\begin{aligned} d_{\lambda K} &= (\text{multiplicity of } L(K) \text{ in a composition series of } \Delta_0(\lambda)) \\ &= \mathrm{Card}\{w \in \mathcal{F}^\lambda \mid D_\ell(w) = K\}. \end{aligned}$$

and

$$c_{JK} = \sum_\lambda d_{\lambda J} d_{\lambda K}, \qquad \text{or, in matrix notation,} \qquad C = D^t D,$$

where

$$\begin{aligned} &C = (c_{JK}), \text{ is the Cartan matrix, } \quad \text{and} \\ &D = (d_{\lambda K}), \quad \text{is the decomposition matrix.} \end{aligned}$$

This can be checked directly when H is the dihedral Hecke algebra by noting that the irreducible representations of H given in (???) specialize at $q=0$. They are given explicitly by

$$\rho(T_1) = \begin{pmatrix} \frac{-(1+\xi^k)}{\xi^k-\xi^{-k}} & \frac{-(1+\xi^k)(1+\xi^k)}{(\xi^k-\xi^{-k})^2} \\ \xi^{-k} & \frac{1+\xi^{-k}}{\xi^k-\xi^{-k}} \end{pmatrix}, \qquad \rho(T_2) = \begin{pmatrix} \frac{-(1+\xi^k)}{\xi^k-\xi^{-k}} & \frac{-(1+\xi^{-k})(1+\xi^{-k})}{(\xi^k-\xi^{-k})^2} \\ \xi^{-k} & \frac{1+\xi^{-k}}{\xi^k-\xi^{-k}} \end{pmatrix}.$$

Then, in this representation,

$$-T_2 \quad \text{projects onto} \quad v_2 = \begin{pmatrix} \frac{1+\xi^k}{\xi^k-\xi^{-k}} \\ 1 \end{pmatrix},$$

and $T_1 v_2 = 0$. Similarly,

$$-T_1 \quad \text{projects onto} \quad v_1 = \begin{pmatrix} \frac{1+\xi^k}{\xi^k-\xi^{-k}} \\ -\xi^{-k} \end{pmatrix},$$

and

$$T_2 v_1 = \frac{(1-xi^{-2k})(1+\xi^k)}{\xi^k-\xi^{-k}} \begin{pmatrix} \frac{-(1+\xi^k)}{\xi^k-\xi^{-k}} \\ 1 \end{pmatrix}$$

which is a multiple of v_2 (equal to 0 mod v_2). From this one deduces the decomposition matrices as in (???).

1.5 Cell modules

Let $\Delta(\lambda)$ be the cell module of $H(q)$ indexed by the cell λ and let $\Delta_0(\lambda)$ be its specialization at $q=0$. The module $\Delta_0(\lambda)$ has basis

$$\{\overline{C_w} \mid w \in \mathcal{F}^\lambda\} \qquad \text{where } \mathcal{F}^\lambda \text{ is the left cell in } W \text{ indexed by } \lambda.$$

By [KL, 2.3(a)-2.3(c)]

$$T_{s_i} C_w = \begin{cases} -C_w, & \text{if } s_i w < w, \\ qC_w + q^{\frac12} C_{s_i w} + q^{\frac12} \sum_{s_i z < z} \mu(z,w) C_z, & \text{if } s_i w > w, \end{cases}$$

where $\mu(z,w)$ is the coefficient of $q^{\frac12(\ell(w)-\ell(z)-1)}$ in the Kazhdan-Lusztig polynomial $P_{z,w}$. The *cell decomposition numbers*

$$\begin{aligned} \kappa_{\lambda K} &= (\text{multiplicity of } L_0(K) \text{ in a composition series of } \Delta_0(\lambda)) \\ &= \mathrm{Card}\{w \in \mathcal{F}^\lambda \mid D_\ell(w) = K\}. \end{aligned}$$

The decomposition of W into left and right cells

$$\begin{array}{ccc} W & \stackrel{\sim}{\longleftrightarrow} & (\text{left cell}, \text{right cell}) \\ w & \longmapsto & (P(w), Q(w)) \end{array}$$

gives the formula????

$$c_{JK} = \sum_\lambda \kappa_{\lambda J} \kappa_{\lambda K}, \qquad \big(C = \kappa^t \kappa\big),$$

where

$$\begin{aligned} C &= (c_{JK}), \text{ is the Cartan matrix,} \quad \text{and} \\ \kappa &= (\kappa_{\lambda K}), \quad \text{is the cell decomposition matrix.} \end{aligned}$$

For dihedral groups, $I_2(m)$, the left cells are

$$\begin{aligned} \Gamma^{\emptyset} = \{1\}, \quad \Gamma^1 &= \{s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1, \ \ldots, \ \underbrace{\cdots s_1s_2s_1}_{m \text{ factors}}\}, \\ \Gamma^2 &= \{s_2, s_1s_2, s_2s_1s_2, s_1s_2s_1s_2, \ \ldots, \ \underbrace{\cdots s_2s_1s_2}_{m \text{ factors}}\}, \quad \Gamma^{12} = \{w_0\} \end{aligned}$$

and at $q = 0$ the cell representations have matrices

$$\Delta_0^{\emptyset}(T_1) = (-1), \qquad \Delta_0^{\emptyset}(T_2) = (-1),$$

$$\Delta_0^1(T_1) = \begin{pmatrix} 0 &&&& \\ & -1 &&& \\ && 0 && \\ &&& -1 & \\ &&&& \ddots \end{pmatrix} \qquad \Delta_0^1(T_2) = \begin{pmatrix} -1 &&&& \\ & 0 &&& \\ && -1 && \\ &&& 0 & \\ &&&& \ddots \end{pmatrix}$$

are $(m-1) \times (m-1)$ matrices with rows and columns indexed by $C_1, C_{21}, C_{121}, \ldots$,

$$\Delta_0^2(T_1) = \begin{pmatrix} -1 &&&& \\ & 0 &&& \\ && -1 && \\ &&& 0 & \\ &&&& \ddots \end{pmatrix} \qquad \Delta_0^2(T_2) = \begin{pmatrix} 0 &&&& \\ & -1 &&& \\ && 0 && \\ &&& -1 & \\ &&&& \ddots \end{pmatrix}$$

are $(m-1) \times (m-1)$ matrices with rows and columns indexed by $C_2, C_{12}, C_{212}, \ldots$, and

$$\Delta_0^{\{1,2\}}(T_1) = (0), \qquad \Delta_0^{\{1,2\}}(T_2) = (0).$$

Hence

$$\begin{aligned} \Delta_0(\emptyset) &\cong L(\emptyset), \\ \Delta_0^{\{1\}} &\cong \begin{cases} \frac{m-1}{2} L(\{1\}) \oplus \frac{m-1}{2} L(\{2\}), & \text{if } m \text{ is odd,} \\ \frac{m}{2} L(\{1\}) \oplus \frac{m-2}{2} L(\{2\}), & \text{if } m \text{ is even,} \end{cases} \\ \Delta_0^{\{2\}} &\cong \begin{cases} \frac{m-1}{2} L(\{1\}) \oplus \frac{m-1}{2} L(\{2\}), & \text{if } m \text{ is odd,} \\ \frac{m-2}{2} L(\{1\}) \oplus \frac{m-1}{2} L(\{2\}), & \text{if } m \text{ is even,} \end{cases} \\ \Delta_0^{\{1,2\}} &\cong L(\{1,2\}). \end{aligned}$$

So the decomposition matrix for the cell representations is

$$\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m-1}{2} & \frac{m-1}{2} & 0 \\ 0 & \frac{m-1}{2} & \frac{m-1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{when } m \text{ is odd,}$$

and

$$\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m}{2} & \frac{m}{2}-1 & 0 \\ 0 & \frac{m}{2}-1 & \frac{m}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{when } m \text{ is even.}$$

Thus $\kappa = C$ for dihedral Hecke algebras at $q = 0$ (but the cell modules at $q = 0$ are *not* the projective indecomposables–the cell modules at $q = 0$ are semisimple!).

1.6 Examples

For A_2, with rows indexed by the partitions $(3), (21), (1^3)$ and columns indexed by the subsets $\emptyset, \{1\}, \{2\}, \{1, 2\}$,

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$D^tD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For B_2, with rows indexed by pairs of partitions

$$((2), \emptyset), \quad ((1^2), \emptyset), \quad ((1), (1)), \quad (\emptyset, (2)), \quad (\emptyset, (1^2))$$

and columns indexed by the subsets $\emptyset, \{1\}, \{2\}, \{1, 2\}$,

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{picture of Bruhat graph}$$

and

$$D^tD = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $I_2(5)$, with rows indexed by

$$\chi_1^+, \quad \chi_2^1, \quad \chi_2^2, \quad \chi_1^-,$$

and columns indexed by the subsets $\emptyset, \{1\}, \{2\}, \{1, 2\}$,

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{picture of Bruhat graph}$$

and

$$D^t D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For G_2, with rows indexed by pairs of partitions

$$\chi_1^{++}, \quad \chi_1^{+-}, \quad \chi_2^1, \quad \chi_2^2, \quad \chi_1^{-+}, \quad \chi_1^{--},$$

and columns indexed by the subsets $\emptyset, \{1\}, \{2\}, \{1,2\}$,

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{picture of Bruhat graph}$$

and

$$D^t D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, in general,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lfloor \frac{m}{2} \rfloor & \lfloor \frac{m-1}{2} \rfloor & 0 \\ 0 & \lfloor \frac{m-1}{2} \rfloor & \lfloor \frac{m}{2} \rfloor & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{for } I_2(m),$$

If m is odd then

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m-1}{2} & \frac{m-1}{2} & 0 \\ 0 & \frac{m-1}{2} & \frac{m-1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with rows indexed by

$$\chi_1^{+}, \quad \chi_2^1, \quad \chi_2^2, \quad \ldots, \quad \chi_2^{\frac{m-1}{2}}, \quad \chi_1^{-},$$

and columns indexed by the subsets $\emptyset, \{1\}, \{2\}, \{1,2\}$. If m is even then

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m}{2} & \frac{m}{2} - 1 & 0 \\ 0 & \frac{m}{2} - 1 & \frac{m}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with rows indexed by

$$\chi_1^{++}, \chi_1^{+-}, \quad \chi_2^1, \quad \chi_2^2, \quad \ldots, \quad \chi_2^{\frac{m}{2}-1}, \quad \chi_1^{-+}, \quad \chi_1^{--},$$

and columns indexed by the subsets $\emptyset, \{1\}, \{2\}, \{1,2\}$.

2 Representations of dihedral Hecke algebras

The *dihedral Hecke algebra* $H = H_{m,m,2}(p,q)$ can be given by generators T_1, T_2 and relations

$$T_1^2 = (p-1)T_1 + p, \quad T_2^2 = (q-1)T_2 + q, \quad \text{and} \quad \underbrace{T_1T_2T_1\cdots}_{m \text{ factors}} = \underbrace{T_2T_1T_2\cdots}_{m \text{ factors}}.$$

The braid relations give that $(T_1T_2)^m \in Z(H)$.

If $\rho\colon H \to M_2(\mathbb{C})$ is an irreducible representation of H then

$$\rho(T_1T_2)^m = c\cdot \mathrm{id} \qquad \text{and} \qquad c^2 = \det(T_1T_2)^m = (-p)^m(-q)^m = (pq)^m.$$

So $c = \pm(pq)^{m/2}$ and, when $p=q=1$, $\rho(s_1s_2)^m = 1$. So $c = (pq)^m$. So

$$\begin{aligned}&\rho(T_1T_2p^{-\frac{1}{2}}q^{-\frac{1}{2}}) \ \text{ has root of unity eigenvalues, and}\\ &\det(\rho(T_1T_2p^{-\frac{1}{2}}q^{-\frac{1}{2}})) = (-p)(-q)p^{-1}q^{-1} = 1,\end{aligned}$$

since, because ρ is irreducible $\rho(T_1)$ must have two distinct eigenvalues which must be p and -1 because of the equation $(T_1-p)(T_1+1)=0$, and similarly $\rho(T_1)$ has eigenvalues q and -1. So, with appropriate choice of basis

$$\rho(T_1T_2) = p^{\frac{1}{2}}q^{\frac{1}{2}}\begin{pmatrix}\xi^k & 0\\ 0 & \xi^{-k}\end{pmatrix}, \qquad 0 < k < \lfloor \tfrac{m}{2} \rfloor.$$

If

$$\rho(T_2) = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \qquad \text{with } a+d = q-1 \text{ and } ad-bc = -q,$$

then we may assume that

$$\rho(T_2) = \begin{pmatrix} a & q+ad\\ 1 & d\end{pmatrix},$$

in which case

$$\rho(T_1) = \rho(T_1T_2T_2^{-1}) = p^{\frac{1}{2}}q^{\frac{1}{2}}\begin{pmatrix}\xi^k & 0\\ 0 & \xi^{-k}\end{pmatrix}\begin{pmatrix}-d & q+ad\\ 1 & -a\end{pmatrix}q^{-1} = p^{\frac{1}{2}}q^{-\frac{1}{2}}\begin{pmatrix}-d\xi^k & (q+ad)\xi^k\\ \xi^{-k} & -a\xi^{-k}\end{pmatrix}$$

Then $a+d = q-1$ and $p^{\frac{1}{2}}q^{-\frac{1}{2}}(-d\xi^k - a\xi^{-k}) = p-1$. Solving for a and d gives

$$a = \frac{(q-1)\xi^k + (p-1)p^{-\frac{1}{2}}q^{\frac{1}{2}}}{\xi^k - \xi^{-k}} \qquad \text{and} \qquad d = \frac{(q-1)\xi^{-k} + (p-1)p^{-\frac{1}{2}}q^{\frac{1}{2}}}{\xi^{-k} - \xi^k}$$

If $p=q$ then

$$a = \frac{(q-1)(\xi^k+1)}{\xi^k - xi^{-k}} \qquad \text{and} \qquad d = \frac{(q-1)(\xi^{-k}+1)}{\xi^{-k}-\xi^k}.$$

References

[HR] T. Halverson and A. Ram, *Partition algebras*, European J. Combinatorics **26** (2005), 869–921.

[Ko1] M. Kosuda, *Irreducible representations of the party algebra*, preprint 2004.

[Ko2] M. Kosuda, *Characterization of the party algebras* Ryukyu Math. J. **13** (2003), 199–228.

[Ta] K. Tanabe, *On the centralizer algebra of the unitary reflection group* $G(m,p,n)$, Nagoya Math. J. **148** (1997), 113-126.

[Dr1] V.G. Drinfel'd, *A new realization of Yangians and quantized affine algebras*, Soviet Math. Dokl. **36** No, 2 (1998), 212–216.