Quasitrangular Hopf algebras and the quantum double

1. Quasitriangular Hopf algebras

1.1 Let A be a Hopf algebra with coproduct Δ and antipode S. Let $\sigma\colon A\otimes A\to A\otimes A$ be the map given by $\sigma(a\otimes b)=b\otimes a$ for all $a,b\in A$. Define Δ' to be the opposite coproduct given by

$$\Delta'=\sigma\circ\Delta.$$

Then A with coproduct Δ' and antipode S^{-1} is also a Hopf algebra. This follows by applying S^{-1} to the defining relation for the antipode

$$\sum_a a_{(1)}S(a_{(2)})=\sum_a S(a_{(1)})a_{(2)}=\varepsilon(a),$$

for all $a\in A$ and using the fact that S (and therefore S^{-1}) is an antihomomorphism.

1.2 A pair (A,R) consisting of a Hopf algebra A and an invertible element $R\in A\otimes A$ is called *quasitriangular* if

a) $\Delta'(a)=R\Delta(a)R^{-1}$, for all $a\in A$,
b) $(\Delta\otimes id)(R)=R^{13}R^{23}$,
c) $(id\otimes\Delta)(R)=R^{13}R^{12}$,

where, if $R=\sum_i a_i\otimes b_i$ then

$$R^{12}=\sum_i a_i\otimes b_i\otimes 1,\quad R^{13}=\sum_i a_i\otimes 1\otimes b_i,\quad R^{23}=\sum_i 1\otimes a_i\otimes b_i\otimes 1,\quad \text{etc.}$$

(1.3) Theorem. ([D1] Prop. 3.1) *If (A,R) is a quasitriangular Hopf algebra then*

a) $R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$.
b) $\check R_{12}\check R_{23}\check R_{12}=\check R_{23}\check R_{12}\check R_{23}$,
where $\check R_{ij}=\sigma\circ L_{R^{ij}}\in \mathrm{End}(A\otimes A)$, and $\sigma, L_R\in\mathrm{End}(A\otimes A)$ are given by $\sigma(a\otimes b)=b\otimes a$ and left multiplication by R respectively.
c) $(\varepsilon\otimes id)(R)=1=(id\otimes\varepsilon)(R)$.
d) $(S\otimes id)(R)=(\mathrm{id}\otimes S^{-1})(R)=R^{-1}$.
e) $(S\otimes S)(R)=R$.

Proof. a)

$$\begin{aligned}R^{12}R^{13}R^{23}&=R^{12}(\Delta\otimes id)(R)\quad\text{by (1.2b)}\\&=(\Delta'\otimes id)(R)R^{12}\quad\text{by (1.2a)}\\&=R^{23}R^{13}R^{12}.\end{aligned}$$

b)

$$\begin{aligned}\check R_{12}\check R_{23}\check R_{12}&=\sigma^{12}L_{R^{12}}\sigma^{23}L_{R^{23}}\sigma^{12}L_{R^{12}}\\&=\underbrace{\sigma^{12}\sigma^{23}\sigma^{12}}\,\underbrace{\sigma^{12}\sigma^{23}L_{R^{12}}\sigma^{23}\sigma^{12}}\,\underbrace{\sigma^{12}L_{R^{23}}\sigma^{12}}\,L_{R^{12}}\\&=\sigma^{13}L_{R^{23}}L_{R^{13}}L_{R^{12}},\end{aligned}$$

and

$$\begin{aligned}\check R_{23}\check R_{12}\check R_{23}&=\sigma^{23}L_{R^{23}}\sigma^{12}L_{R^{12}}\sigma^{23}L_{R^{23}}\\&=\underbrace{\sigma^{23}\sigma^{12}\sigma^{23}}\,\underbrace{\sigma^{23}\sigma^{12}L_{R^{23}}\sigma^{12}\sigma^{23}}\,\underbrace{\sigma^{23}L_{R^{12}}\sigma^{23}}\,L_{R^{23}}\\&=\sigma^{13}L_{R^{12}}L_{R^{13}}L_{R^{23}},\end{aligned}$$

c) By (1.2b)

$$R = (id \otimes id)(R) = (\varepsilon \otimes id \otimes id)(\Delta \otimes id)(R) = (\varepsilon \otimes id \otimes id)R^{13}R^{23} = (\varepsilon \otimes id)(R) \cdot R.$$

Thus $(\varepsilon \otimes id)(R) = 1$. Similarly, by (1.2c),

$$R = (id \otimes id)(R) = (id \otimes id \otimes \varepsilon)(id \otimes \Delta)(R) = (id \otimes id \otimes \varepsilon)R^{13}R^{23} = (id \otimes \varepsilon)(R) \cdot R.$$

Thus $(id \otimes \varepsilon)(R) = 1$.

d)

$$\begin{aligned} R \cdot (S \otimes id)(R) &= (m \otimes id)(id \otimes S \otimes id)(R^{13}R^{23}) \\ &= (m \otimes id)(id \otimes S \otimes id)(\Delta \otimes id)(R) \\ &= (\varepsilon \otimes id)(R) = 1. \end{aligned}$$

So $(S \otimes id)(R) = R^{-1}$. Let A^{opp} be the Hopf algebra which is the same as A except with the opposite comultiplication and with antipode S^{-1}. It is clear from the defining relations of a quasitriangular Hopf algebra that (A^{opp}, R^{21}) is also a quasitriangular Hopf algebra. Thus, it follows by applying the identity already proved to (A^{opp}, R^{21}) that

$$(S^{-1} \otimes id)(R^{21}) = (R^{-1})^{21}$$

which is equivalent to $(id \otimes S^{-1})(R) = R^{-1}$.

e) This follows by letting $(id \otimes S)$ act on both sides of the equation $(id \otimes S^{-1})(R) = (S \otimes id)(R)$ from d). $\square$

2. The Quantum double

(2.1) Theorem. ([D1] §13) *Let A be a finite dimensional Hopf algebra and let A^{*opp} denote the Hopf algebra A^* except with the opposite comultiplication. Then there exists a unique quasitriangular Hopf algebra $(D(A), R)$ such that*

1) $D(A)$ contains A and A^{*opp} as Hopf subalgebras.

2) R is the image of the canonical element of $A \otimes A^{*opp}$ under $A \otimes A^{*opp} \to D(A) \otimes D(A)$, i.e. if e_i is a basis of A and e^i is the dual basis in A^{*opp} then

$$R = \sum e_i \otimes e^i \in D(A) \otimes D(A).$$

3) The linear map

$$\begin{array}{ccc} A \otimes A^{*opp} & \to & D(A) \\ a \otimes b & \mapsto & ab \end{array}$$

is bijective.

2.2 *Remark.* If A is infinite dimensional then one may be able to apply the theorem if there is a suitable way of completing the tensor product $D(A) \otimes D(A)$ so that the element $R = \sum e_i \otimes e^i$ is a well defined element of the completion $D(A)\hat{\otimes}D(A)$.

Proof of Theorem 2.1.

2.3 Let the algebra A be the Hopf algebra with basis $\{e_r\}$ and multiplication, comultiplication, and skew antipode given by

$$\begin{aligned} e_r e_s &= \sum_t m_{rs}^t e_t, \\ \Delta(e_t) &= \sum_{r,s} \mu_t^{rs} e_r \otimes e_s, \\ \sigma(e_t) &= \sum_r \sigma_t^r e_r. \end{aligned}$$

The unit and counit will be given by $1 = \sum_t E^t e_t$, and $\varepsilon(e_r) = \varepsilon_r$ respectively. Recall that the skew antipode is the inverse S^{-1} of the antipode of A and is the the antipode for the Hopf algebra A^{opp} which is the same as the algebra A except with the opposite comultiplication.

2.4 The algebra A^{*opp} has basis $\{e^r\}$ which is dual to the basis $\{e_r\}$ of A and has multiplication and comultiplication given by

$$\begin{aligned} e^r e^s &= \sum_t \mu_t^{rs} e_t, \\ \Delta(e^t) &= \sum_{r,s} m_{rs}^t e^s \otimes e^r. \end{aligned}$$

Then the algebra $A \otimes A^{*opp}$ has basis $\{e^r e_s\}$ and has multiplication given by

$$(e^r e_s \otimes e^p e_q)(e^k e_l \otimes e^m e_n) = (e^r e_s e^k e_l \otimes e^p e_q e^m e_n), \qquad (*)$$

and comultiplication given by

$$\begin{aligned} \Delta(e^r e_s) &= \Delta(e^r)\Delta(e_s) \\ &= \left(\sum_{u,v} m_{uv}^r e^v \otimes e^u \right) \left(\sum_{p,q} \mu_s^{pq} e_p \otimes e_q \right) \\ &= \sum_{u,v,p,q} m_{uv}^r \mu_s^{pq} e^v e_p \otimes e^u e_q. \end{aligned}$$

Alternatively, we could have chosen to use the basis $\{e_r e^s\}$ instead of the basis $\{e^r e_s\}$. It is clear from $(*)$ that we need to describe a product $e_s e^k$ in terms of the basis $e^p e_q$ in order to completely describe the multiplication in $A \otimes A^{*opp}$.

2.5 We shall use the condition $R\Delta'(a)R^{-1} = \Delta(a)$ to determine the formula for a product $e_s e^k$ in terms of the basis $e^p e_q$. The relation is

$$e_r e^s = \sum_{\alpha,\beta,\gamma,\delta,p} \mu_r^{\gamma\beta\alpha} \sigma_\alpha^p m_{p\delta\gamma}^s e^\delta e_\beta.$$

This relation is derived as follows.

$$\begin{aligned}
\langle e^v e_b, e_j e^l \rangle &= \langle e^v e_b, m \circ \sigma(e^l \otimes e_j) \rangle \\
&= \langle \sigma \circ \Delta(e^v e_b), e^l \otimes e_j \rangle \\
&= \langle \Delta'(e^v e_b), e^l \otimes e_j \rangle \\
&= \langle R\Delta(e^v e_b) R^{-1}, e^l \otimes e_j \rangle \\
&= \langle R\Delta(e^v e_b)((\mathrm{id} \otimes S^{-1})(R), e^l \otimes e_j \rangle \quad \text{by (1.3e)} \\
&= \langle R \otimes \Delta(e^v e_b) \otimes ((\mathrm{id} \otimes S^{-1})(R), (\Delta^\otimes)^2(e^l \otimes e_j) \rangle
\end{aligned}$$

Let us expand the left hand factor of this inner product.

$$\begin{aligned}
R \otimes \Delta(e^v e_b) \otimes (id \otimes S^{-1})(R) &= \sum_{k,p} e_k \otimes e^k \otimes \Delta(e^v e_b) \otimes (id \otimes S^{-1})(e_p \otimes e^p) \\
&= \sum_{\substack{k,p,q \\ r,s,t,u}} e_k \otimes e^k \otimes m_{sr}^v \mu_b^{ut} e^r e_u \otimes e^s e_t \otimes e_p \otimes \sigma_q^p e^q
\end{aligned} \qquad (2.5a)$$

The right hand factor of the inner product expands in the form

$$\begin{aligned}
(\Delta^\otimes)^2(e^l \otimes e_j) &= (\Delta^{otimes} \otimes id^\otimes) \circ \Delta^\otimes(e^l \otimes e_j) \\
&= (\Delta^{otimes} \otimes id^\otimes) \left(\sum_{x,y,w,z} m_{xy}^l \mu_j^{wz} e^y \otimes e_w \otimes e^x \otimes e_z \right) \\
&= \sum_{\substack{x,y,w,z \\ m,n,c,d}} m_{mn}^y \mu_w^{cd} m_{xy}^l \mu_j^{wz} e^n \otimes e_c \otimes e^m \otimes e_d \otimes e^x \otimes e_z \\
&= \sum_{\substack{m,n,x \\ c,d,z}} m_{xmn}^l \mu_j^{cdz} e^n \otimes e_c \otimes e^m \otimes e_d \otimes e^x \otimes e_z
\end{aligned}$$

Now let us evaluate the inner product. The inner product picks out only the terms when

$$k = n, k = c, v = m, b = d, p = x, q = z,$$

and this term appears with coefficient

$$m_{xmn}^l \mu_j^{cdz} \sigma_q^p = m_{pvk}^l \mu_j^{kbq} \sigma_q^p.$$

It follows that

$$\langle e^v e_b, e_j e^l \rangle = \sum_{p,q,k} = m_{pvk}^l \mu_j^{kbq} \sigma_q^p.$$

The multiplication rule follows.

2.6 We shall need the following calculation in our proof that $D(A)$ is quasitriangular. We shall need the identities in §4 of the notes on co-Poisson Hopf algebras.

$$\begin{aligned}
\sum_{\gamma,\alpha,p,s} \mu_a^{\gamma\beta\alpha s}\sigma_\alpha^p m_{sp\delta\gamma}^v &= \sum_{\gamma.\alpha,n,k,s,p} \mu_a^{\gamma\beta n}\mu_n^{\alpha s}\sigma_\alpha^p m_{sp}^k m_{k\delta\gamma}^v \qquad \text{by 4.1 and 4.4} \\
&= \sum_{\gamma,\alpha,n,k} \mu_a^{\gamma\beta n}\varepsilon_n E^k m_{k\delta\gamma}^v \qquad \text{by 4.11} \\
&= \sum_{\gamma,\alpha,n,k} \mu_a^{\gamma\beta n}\varepsilon_n \delta_{\delta m} m_{m\gamma}^v \qquad \text{by 4.2} \\
&= \sum_{\gamma,n} \mu_a^{\gamma\beta n}\varepsilon_n m_{\delta\gamma}^v \\
&= \sum_{\gamma,n,k} \mu_a^{\gamma k}\mu_k^{\beta n}\varepsilon_n m_{\delta\gamma}^v \qquad \text{by 4.4} \\
&= \sum_{\gamma,k} \mu_a^{\gamma k}\delta_{k\beta} m_{\delta\gamma}^v \qquad \text{by 4.5} \\
&= \sum_{\gamma} \mu_a^{\gamma\beta} m_{\delta\gamma}^v
\end{aligned}$$

2.7 Now we prove that $A \otimes A^{*opp}$ satisfies the first condition (1.2a) for a quasitriangular Hopf algebra.

$$\begin{aligned}
((\sigma \circ \Delta)(e^v e_b))R &= \sum_{\delta,\gamma,r,s,m} m_{\delta\gamma}^v \mu_b^{rs}(e^\delta e_s \otimes e^\gamma e_r)(e_m \otimes e^m) \\
&= \sum_{\delta,\gamma,r,s,m} m_{\delta\gamma}^v \mu_b^{rs}(e^\delta e_s e_m \otimes e^\gamma e_r e^m) \\
&= \sum_{\delta,\gamma,r,s,m,\lambda} m_{\delta\gamma}^v \mu_b^{rs} m_{sm}^\lambda (e^\delta e_\lambda \otimes e^\gamma e_r e^m) \\
&= \sum_{\substack{\delta,\gamma,r,s,m,\lambda \\ u,t,\alpha,p,\beta}} m_{\delta\gamma}^v \mu_b^{rs}\mu_r^{ut\alpha}\sigma_\alpha^p m_{p\beta u}^m m_{sm}^\lambda (e^\delta e_\lambda \otimes e^\gamma e^\beta e_t) \\
&= \sum_{\substack{\delta,\gamma,r,s,m,\lambda \\ u,t,\alpha,p,\beta,a}} m_{\delta\gamma}^v \mu_a^{\gamma\beta}\mu_b^{rs}\mu_r^{ut\alpha}\sigma_\alpha^p m_{p\beta u}^m m_{sm}^\lambda (e^\delta e_\lambda \otimes e^a e_t) \\
&= \sum_{\substack{\delta,\gamma,s,m,\lambda \\ u,t,\alpha,p,\beta,a}} m_{\delta\gamma}^v \mu_a^{\gamma\beta}\mu_b^{ut\alpha s}\sigma_\alpha^p m_{p\beta u}^m m_{sm}^\lambda (e^\delta e_\lambda \otimes e^a e_t) \\
&= \sum_{\substack{\delta,\gamma,s,\lambda \\ u,t,\alpha,p,\beta,a}} m_{\delta\gamma}^v \mu_a^{\gamma\beta}\mu_b^{ut\alpha s}\sigma_\alpha^p m_{sp\beta u}^\lambda (e^\delta e_\lambda \otimes e^a e_t) \\
&= \sum_{\delta,\gamma,u,t,\beta,a,\lambda} m_{\delta\gamma}^v \mu_a^{\gamma\beta}\mu_b^{ut} m_{\beta u}^\lambda (e^\delta e_\lambda \otimes e^a e_t)
\end{aligned}$$

A similar calculation on the right hand side gives

$$\begin{aligned}
R\Delta(e^v e_b) &= \sum_{r,s,t,u,m} m^v_{sr}\mu^{ut}_b (e_m \otimes e^m)(e^r e_u \otimes e^s e_t) \\
&= \sum_{r,s,t,u,m} m^v_{sr}\mu^{ut}_b (e_m e^r e_u \otimes e^m e^s e_t) \\
&= \sum_{r,s,t,u,m,a} m^v_{sr}\mu^{ut}_b \mu^{ms}_a (e_m e^r e_u \otimes e^a e_t) \\
&= \sum_{\substack{r,s,t,u,m,a\\ \alpha,\beta,\gamma,\delta,p}} m^v_{sr}\mu^{ut}_b \mu^{ms}_a \mu^{\gamma\beta\alpha}_m \sigma^p_\alpha m^r_{p\delta\gamma} (e^\delta e_\beta e_u \otimes e^a e_t) \\
&= \sum_{\substack{r,s,t,u,m,a\\ \alpha,\beta,\gamma,\delta,p,\lambda}} m^v_{sr}\mu^{ut}_b \mu^{ms}_a \mu^{\gamma\beta\alpha}_m \sigma^p_\alpha m^r_{p\delta\gamma} m^\lambda_{\beta u} (e^\delta e_\lambda \otimes e^a e_t) \\
&= \sum_{\substack{s,t,u,a\\ \alpha,\beta,\gamma,\delta,p,\lambda}} \mu^{ut}_b \mu^{\gamma\beta\alpha s}_a \sigma^p_\alpha m^v_{sp\delta\gamma} m^\lambda_{\beta u} (e^\delta e_\lambda \otimes e^a e_t) \\
&= \sum_{\delta,\gamma,u,t,\beta,a,\lambda} m^v_{\delta\gamma}\mu^{\gamma\beta}_a \mu^{ut}_b m^\lambda_{\beta u} (e^\delta e_\lambda \otimes e^a e_t)
\end{aligned}$$

2.8 It remains to prove the identities $(id \otimes \Delta)(R) = R^{13}R^{12}$ and $(\Delta \otimes id)(R) = R^{13}R^{23}$.

$$\begin{aligned}
(id \otimes \Delta)(R) &= \sum_k e_k \otimes \Delta(e^k) \\
&= \sum_{k,r,s} m^k_{rs} e_k \otimes e^s \otimes e^r \\
&= \sum_{r,s} e_r e_s \otimes e^s \otimes e^r \\
&= \sum_{r,s} (e_r \otimes 1 \otimes e^r)(e_s \otimes e^s \otimes 1) \\
&= R^{13}R^{12}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
(\Delta \otimes id)(R) &= \sum_k \Delta(e_k) \otimes e^k \\
&= \sum_{k,r,s} \mu^{rs}_k e_r \otimes e_s \otimes e^k \\
&= \sum_{r,s} e_r \otimes e_s \otimes e^r e^s \\
&= \sum_{r,s} (e_r \otimes 1 \otimes e^r)(1 \otimes e_s \otimes e^s) \\
&= R^{13}R^{23}.
\end{aligned}$$

This completes the proof of Theorem 2.1.

3. References

The quantum double seems to have appeared first in the following paper.

[D] V.G. Drinfeld, *Quantum Groups*, Vol. 1 of *Proceedings of the International Congress of Mathematicians, Berkeley, California, USA, 1986.* Academic Press, 1987, pp. 798-820.

Some further proofs and hints appear in the following.

[D2] V.G. Drinfel'd, *On almost cocommutative Hopf algebras*, Leningrad Math. J. **1** (1990) 321-342.

[Re] N. Yu. Reshetikhin, Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links I, LOMI preprint no. E-4-87, (1987).