The affine flag variety

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1 Loop groups

1.1 The Iwahori subgroup

A common setup is where $\mathbb{F}$ is the field of fractions of $\mathfrak{o}$, the discrete valuation ring $\mathfrak{o}$ is the ring of integers in $\mathbb{F}$, $\mathfrak{p}$ is the unique maximal ideal in $\mathfrak{o}$ and $k = \mathfrak{o}/\mathfrak{p}$ is the residue field. The favourite examples are

$$\begin{array}{lll} \mathbb{F} = \mathbb{C}((t)) & \mathfrak{o} = \mathbb{C}[[t]] & k = \mathbb{C}, \\ \mathbb{F} = \mathbb{Q}_p & \mathfrak{o} = \mathbb{Z}_p & k = \mathbb{F}_p, \\ \mathbb{F} = \mathbb{F}_q((t)) & \mathfrak{o} = \mathbb{F}_q[[t]] & k = \mathbb{F}_q, \end{array}$$

where $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Z}_p$ is the ring of p-adic integers, and $\mathbb{F}_q$ is the finite field with q elements.

The diagram

$$\begin{tikzcd}
 & & & G & = & G(\mathbb{F}) & \\
\mathbb{F} & & & \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \text{gives} & K & = & G(\mathfrak{o}) \arrow[r, "\Phi"] & G(k) \\
\mathfrak{o} \arrow[r] & k = \mathfrak{o}/\mathfrak{p} & & \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & \rotatebox[origin=c]{90}{$\subseteq$} \\
 & & & I & = & \Phi^{-1}(B(k)) \arrow[r] & B(k)\,,
\end{tikzcd}$$

where

$$B(k) \text{ is the group generated by } x_\alpha(c) \text{ and } h_{\lambda^\vee}(d)$$

for $\alpha \in R^+$, $\lambda^\vee \in P^\vee$, $c \in k$, $d \in k^\times$. Then I is the *Iwahori subgroup* of G,

$$G(k)/B(k) \text{ is the } \textit{flag variety,}$$

$$G/I \text{ is the } \textit{affine flag variety,} \qquad \text{and} \qquad G/K \text{ is the } \textit{loop Grassmanian.}$$

If t is the generator of the maximal ideal $\mathfrak{p}$,

$$\begin{aligned} \widetilde{N} &= \langle n_\alpha, h_{\lambda^\vee}(d), h_{\lambda^\vee}(t) \mid \alpha \in R^+, \lambda^\vee \in P^\vee, c \in k^\times \rangle, \\ H &= \langle h_{\lambda^\vee}(d) \mid \lambda^\vee \in P^\vee, d \in k^\times \rangle, \qquad \text{and} \\ \widetilde{W} &= W \ltimes P^\vee = \{ t_{\lambda^\vee} w \mid \lambda^\vee \in P^\vee, w \in W \}, \end{aligned}$$

then H is a normal subgroup of $\widetilde{N}$ and the map

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \widetilde{N}/H \\ s_\alpha & \longrightarrow & n_\alpha H \\ t_{\lambda^\vee} & \longrightarrow & h_{\lambda^\vee}(t^{-1})H \end{array} \qquad \text{is an isomorphism.}$$

Since $H \subseteq I$ the coset notation wI makes sense for $w \in \widetilde{W}$.

The following theorem is a consequence of ??? and ???.

Theorem 1.1. *With notations as in (???), (???) and (???)*

$$\begin{array}{cccc} \begin{array}{c}\textit{Bruhat}\\ \textit{decomposition}\end{array} & G(k) = \displaystyle\bigsqcup_{w \in W} BwB & K = \displaystyle\bigsqcup_{w \in W} IwI & \\ \\ \begin{array}{c}\textit{Iwahori}\\ \textit{decomposition}\end{array} & G = \displaystyle\bigsqcup_{w \in \tilde{W}} IwI & G = \displaystyle\bigsqcup_{v \in \tilde{W}} U^- vI & \\ \\ \begin{array}{c}\textit{Cartan}\\ \textit{decomposition}\end{array} & G = \displaystyle\bigsqcup_{\lambda^\vee \in P^\vee_+} Kt_{\lambda^\vee}K & G = \displaystyle\bigsqcup_{\mu^\vee \in P^\vee} U^- t_{\mu^\vee} K & \begin{array}{c}\textit{Iwasawa}\\ \textit{decomposition}\end{array} \end{array}$$

1.2 G/I points in IwI

Let $\tilde{R} = R + \mathbb{Z}\delta$ and define

$$x_{\alpha+j\delta}(c) = x_\alpha(ct^j) \quad \text{and} \quad h_{(\alpha+j\delta)^\vee}(d) = h_{\alpha^\vee}(d),$$

for $\alpha + j\delta \in \tilde{R}$, $c \in k$ and $d \in k^\times$. The action of $\widetilde{W}$ on $\tilde{R}$ is determined by

$$wx_{\alpha+j\delta}(c)w^{-1} = x_{w(\alpha+j\delta)}(c) \qquad \text{mod } H.$$

The analog of R^+ for G is

$$\begin{aligned} \tilde{R}^I &= \{\alpha + j\delta \mid x_{\alpha+j\delta}(c) \in I \text{ for } c \in k\} \\ &= \{\alpha + j\delta \mid \alpha \in R^+, j \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\alpha + j\delta \mid \alpha \in R^+, j \in \mathbb{Z}_{>0}\}. \end{aligned}$$

and the *length* of $w \in \widetilde{W}$ is

$$\ell(w) = \mathrm{Card}(R(w)) \qquad \text{where} \quad R(w) = \{\alpha + j\delta \in \tilde{R}^I \mid w(\alpha + j\delta) \notin \tilde{R}^I\}.$$

The *simple reflections* $s_0, \ldots, s_n$ are the elements of length 1 in $\widetilde{W}$ and the *simple roots* $\alpha_0, \ldots, \alpha_n$ are determined by

$$R(s_i) = \{\alpha_i\} \qquad \text{and we define} \qquad \Omega = \{\gamma \in \widetilde{W} \mid \ell(\gamma) = 0\}.$$

If $\ell(\gamma) = 0$ then

$$I\gamma I = \gamma I_\gamma I = \gamma I.$$

Since

$$Is_jI = s_jI^{s_j}I = s_j\mathfrak{X}_{-\alpha_j}I = \mathfrak{X}_{\alpha_j}s_jI,$$

$\{x_j(c)s_jI \mid c \in k\} \subseteq Is_jI$. If $x_j(c')s_jI = x_j(c)s_jI$ then $s_jx_j(c'-c)s_j \in I$ and $x_{-\alpha_j}(c'-c) \in I$ so that $c'-c=0$ and $c'=c$. Thus

$$\{x_{\alpha_j}(c)s_j \mid c \in k\} \quad \text{is a set of coset representatives of the } I\text{-cosets in } Is_jI.$$

The elements of length 0 and 1 generate $\widetilde{W}$. If $w = \gamma s_{i_1}\cdots s_{i_\ell}$ is a reduced word for w then

$$\gamma x_{i_1}(c_1)s_{i_1}\cdots x_{i_\ell}(c_\ell)s_{i_\ell}I = x_{\beta_1}(c_1)\cdots x_{\beta_\ell}(c_\ell)wI \in IwI,$$

since $\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}\alpha_{i_2}$, …, $\beta_\ell = s_{i_1}\cdots s_{i_{\ell-1}}\alpha_{i_\ell}$ are all in $\tilde{R}^I$.

Proposition 1.2.

(a) Let $w \in \widetilde{W}$. Then

$$IwI\cdot Is_jI = \begin{cases} Iws_jI, & \text{if } ws_j > w,\\ IwI \cup Iws_jI, & \text{if } ws_j < w.\end{cases}$$

(b) $G = \bigsqcup_{w\in\widetilde{W}} IwI$.

(c) Let $\{y_j(c) \mid c \in k\}$ be coset representatives of the I-cosets in Is_jI. Let $w \in W$ and let $w = s_{i_1}\cdots s_{i_\ell}$ be a reduced word for w. Then

$$\{y_{i_1}\cdots y_{i_\ell}(c_\ell) \mid c_1,\ldots,c_\ell \in k\},$$

is a set of coset representatives of the I-cosets in IwI.

Proof. First note that

$$Is_jI\cdot Is_jI = II^{s_j}I = I\mathcal{X}_{-\alpha_j}I = \mathcal{X}_{-\alpha_j}I = I \cup Is_iI.$$

If $ws_j > w$ then $w\alpha_j \in R^+$ and

$$IwI\cdot Is_jI = Iws_jI^{s_j}I = Iws_j\mathcal{X}_{-\alpha_j}I = I\mathcal{X}_{w\alpha_j}ws_jI = Iws_jI,$$

and if $ws_j < w$ then

$$IwI\cdot Is_jI = Iws_jI\cdot Is_jI\cdots Is_jI = Iws_jI\cdot(I\cup Is_jI) = Iws_jI \cup IwI.$$

(b) Assume $IwI = IvI$. If $\ell(w)=0$ then $v \in IwI = wI$ and $w^{-1}v \in I$ giving that $w^{-1}v = 1$, since $I \cap \widetilde{N} = H$. If $\ell(w) > 0$ let s_j be a simple reflection such that $ws_j < w$. Then

$$Iws_jI \subseteq IwI\cdot Is_jI = IvI\cdot Is_jI \quad \text{so that } Iws_jI = Ivs_jI \text{ or } Iws_jI = IvI,$$

since $IvI\cdot Is_jI \subseteq Ivs_jI \cup IvI$. If $Iws_jI = IvI$ then, by induction, $ws_j = v$ and $Iws_jI = IvI = IwI$ so that $ws_j = w$. Since this is impossible, it must be that $Iws_jI = Ivs_jI$ so that, by induction, $ws_j = vs_j$. Hence $w = v$. (c) Since G is generated by I and the $x_\alpha(c)s_\alpha \in Is_\alpha I$. $G = \bigsqcup_{w\in\widetilde{W}} IwI$.

(d) By induction on ℓ, the product $IwI\cdot Is_jI$ contains the points in the sets

$$y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)I\cdot Is_jI = y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)Is_jI = \{y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)y_j(c)I \mid c \in k\}.$$

If $y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)b = y_{i_1}(c_1')\cdots y_{i_\ell}(c_\ell')b'$ then

$$y_{i_1}(c_1')^{-1}y_{i_1}(c_1)y_{i_2}(c_2)\cdots y_{i_\ell}(c_\ell)b = y_{i_2}(c_2')\cdots y_{i_\ell}(c_\ell')b'.$$

If $y_{i_1}(c_1')^{-1}y_{i_1}(c_1) \in Is_{i_1}I\cdot Is_{i_1}I = Is_{i_1}I \cup I$ and, since the right hand side is in $Is_{i_1}wI$ we must have $y_{i_1}(c_1')^{-1}y_{i_1}(c_1) \in I$ so that $y_{i_1}(c_1) \in y_{i_1}(c_1')I$, which forces $c_1 = c_1'$. □

Proposition 1.3. *Let* $q = \mathrm{Card}(k)$. *Then the characteristic functions* $\{T_w \mid w \in \widetilde{W}\}$ *of the double cosets* IwI *are a basis of the Hecke algebra* $\tilde{H} = C_c(I\backslash G/I)$ *and*

$$T_w T_\gamma = T_{w\gamma} \quad \textit{and} \quad T_w T_{s_j} = \begin{cases} T_{ws_j}, & \textit{if } ws_j > w, \\ qT_{ws_j} + (q-1)T_w, & \textit{if } ws_j < w. \end{cases}$$

Proof. The product $IwI \cdot Is_jI$ contains the points

$$\begin{aligned} x_{i_1}(c_1)n_{i_1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}I \cdot Is_jI &= x_{i_1}(c_1)n_{i_1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}Is_jI \\ &= \{x_{i_1}(c_1)n_{i_1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}x_j(c)n_jI \mid c \in k\}. \end{aligned}$$

If $ws_j > w$ then

$$Iws_jI = IwI \cdot Is_jI = \{x_{i_1}(c_1)n_{i_1} \cdots x_{i_\ell}(c_\ell)n_{i_\ell}x_j(c)n_jI \mid c_1, \ldots, c_\ell, c \in k\}.$$

If $ws_j < w$ and $c \neq 0$ then

$$\begin{aligned} &x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_{i_\ell}(c_\ell)s_{i_\ell}x_j(c)s_jI \\ &\quad = x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_1(c_\ell)s_jx_j(c)s_jI \\ &\quad = x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_{\alpha_j}(c_\ell)x_{-\alpha_j}(c)I \\ &\quad = x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_{\alpha_j}(c_\ell + c^{-1})x_{\alpha_j}(-c^{-1})x_{-\alpha_j}(c)x_{\alpha_j}(-c^{-1})I \\ &\quad = x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_{\alpha_j}(c_\ell + c^{-1})s_jI. \end{aligned}$$

If $c = 0$ then

$$x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_1(c_\ell)s_jx_j(0)s_jI = x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}I. \tag{1.1}$$

Hence, in the product

$$\begin{aligned} &x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_j(c_\ell)s_jI \cdot Is_jI \\ &\quad = x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_j(c_\ell)s_jIs_jI \\ &\quad = \{x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_j(c_\ell)s_jx_j(c)s_jI \mid c \in k\}, \end{aligned}$$

the coset

$$x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}x_{\alpha_j}(a_\ell)s_jI,$$

appears once for each choice of $c_\ell \in k$ and $c \in k^\times$ such that $c_\ell + c^{-1} = a_\ell$, a total of $q - 1$ times. The coset

$$x_{i_1}(c_1)s_{i_1} \cdots x_{i_{\ell-1}}(c_{\ell-1})s_{i_{\ell-1}}I,$$

appears once for each choice of $c_\ell \in \mathbb{F}$, a total of q times. □

1.3 G/I points in $U^-vI \cap IwI$

The group

$$U^- = \langle x_{-\alpha}(f) \mid f \in \mathbb{F} \rangle \qquad \text{and} \qquad \tilde{R}^U = R^- + \mathbb{Z}\delta$$

so that $\mathcal{X}_{\alpha+k\delta} \subseteq U^-$ exactly when $\alpha + k\delta \in \tilde{R}^U$. The *periodic orientation* is an orientation of the hyperplanes $H_{\alpha+k\delta}$ such that

(a) 1 is on the postive side of H_α for $\alpha \in R^+$,

(b) $H_{\alpha+k\delta}$ and H_α have parallel orientiations.

Then

$$\alpha+k\delta\in\tilde{R}^U \quad \text{if and only if} \quad \text{the orientation of } H_{\alpha+k\delta} \text{ is } PICTURE.$$

We shall use the identity

$$x_\alpha(f)n_\alpha^{-1}=x_{-\alpha}(f^{-1})x_{-\alpha}(-f)h_{\alpha^\vee}(f) \qquad \text{(main folding law)}$$

to rewrite points of IwI as elements of U^-vI. Suppose that

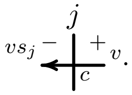
$$x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}=x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_vb$$

Then

$$\begin{aligned}x_{i_1}(c_1)n_{i_1}^{-1}\cdots x_{i_\ell}(c_\ell)n_{i_\ell}^{-1}&=x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_vbx_j(c)n_j^{-1}\\&=x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')n_vx_j(\tilde{c})n_j^{-1}b', \quad \text{since } bx_j(c)n_j^{-1}\in Is_jI.\end{aligned}$$

If $v\alpha_j\in\tilde{R}^U$ then this is equal to

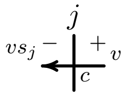
$$x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')x_{v\alpha_j}(\tilde{c})n_{vs_j}b'\in U^-vI\cap Iws_jI.$$

If $v\alpha_j\notin\tilde{R}^U$ and $\tilde{c}\neq0$, 

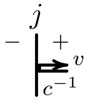
. Then

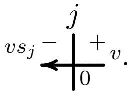
$$\begin{aligned}x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')n_vx_{\alpha_j}(\tilde{c})n_jb'&=x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')n_vx_{-\alpha_j}(\tilde{c}^{-1})x_{\alpha_j}(-\tilde{c})h_{\alpha^\vee}(\tilde{c})b'\\&=x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')n_vx_{-\alpha_j}(\tilde{c}^{-1})b''\\&=x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')x_{\beta_{\ell+1}}(\tilde{c}^{-1})n_vb''\in U^-vI\cap Iws_jI,\end{aligned}$$

where $\beta_{\ell+1}=-v\alpha_j$. So



becomes

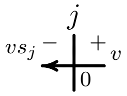


If $v\alpha_j\notin\tilde{R}^U$ and $\tilde{c}=0$, 

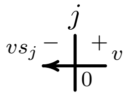
. Then

$$\begin{aligned}x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')n_vx_{\alpha_j}(0)n_jb'&=x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')n_vx_{-\alpha_j}(0)n_jb'\\&=x_{\beta_1}(c_1')\cdots x_{\beta_\ell}(c_\ell')x_{\beta_{\ell+1}}(0)n_{vs_j}b'\in U^-vs_jI\cap Iws_jI,\end{aligned}$$

where $\beta_{\ell+1}=-v\alpha_j$. So

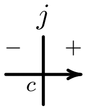


becomes

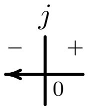


Case 2: $ws_j<w$. This should follow from the computation in 5.1.

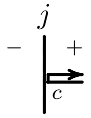
A labeled step of type j is



with $c\in k$, or



or



with $c\in k^\times$,

If $w\in\widetilde{W}$ and $\vec{w}=s_{i_1}\cdots s_{i_\ell}$ be a minimal length walk to w define, for each $v\in\widetilde{W}$,

$$\mathcal{P}(\vec{w})_v=\left\{\begin{matrix}\text{labeled folded paths } p \text{ of type } \vec{w}\\ \text{which end in } v\end{matrix}\right\}.$$

Theorem 1.4. *If $(c_1, \dots, c_\ell) \in B(\vec{w})_v$ then, for some $c_1', \dots, c_\ell' \in k$,*

$$x_{\beta_1}(c_1)x_{\beta_2}(c_2)\cdots x_{\beta_\ell}(c_\ell)wI = x_{i_1}(c_1)s_{i_1}\cdots x_{i_\ell}(c_\ell)s_{i_\ell}I = x_{\gamma_1}(c_1')\cdots x_{\gamma_\ell}(c_\ell')vI$$

$$\text{where} \qquad \beta_k = z\alpha_j, \qquad \text{if the kth step of p is } {}^{z}\overset{j}{\xrightarrow{\;\;|\;\;}}{}^{zs_j},$$

and

$$\gamma_k = \begin{cases} z\alpha_j, & \text{if the kth step of $\Phi(p)$ is } {}^{z}\,{}^{-}\overset{j}{\xrightarrow{\;\;|\;\;}}{}^{+}\,{}^{zs_j}, \\[2em] -z\alpha_j, & \text{if the kth step of $\Phi(p)$ is } {}^{zs_j}\,{}^{-}\overset{j}{\xleftarrow{\;\;|\;\;}}{}^{+}\,{}^{z} \text{ or } {}^{-}\overset{j}{\Big|\!\leftrightarrows}{}^{+}\,{}^{z}. \end{cases}$$

(b) The bijection in (???) restricts to a bijection

$$\mathcal{P}(\vec{w})_v \longleftrightarrow U^-vI \cap IwI \qquad \text{and} \qquad G = \bigsqcup_{v\in\widetilde{W}} U^-vI.$$

Proposition 1.5. *Let $q = \mathrm{Card}(k)$. Then the characteristic functions $\{X_v \mid v \in \widetilde{W}\}$ of the double cosets U^-vI are a basis of the right $\tilde{H}$-module $C_c(U^-\backslash G/I)$,*

$$X_vT_\gamma = X_{w\gamma}, \quad \text{and} \quad X_vT_{s_j} = \begin{cases} X_{vs_j} & \text{if} \quad {}^{-}_{v}\Big|^{+}_{vs_j}, \\[1em] qX_{vs_j} + (q-1)X_v, & \text{if} \quad {}^{-}_{vs_j}\Big|^{+}_{v}. \end{cases}$$

1.4 The closure order

1.5 The loop Grassmanian G/K

Let $\nu^\vee \in P^\vee$. The $\nu^\vee$-*hexagon* is the set of alcoves in

$$t_{\nu^\vee}W = PICTURE$$

The *ending hexagon* $eh(p)$ and the *final direction* $\varphi(p)$ of $p \in \mathcal{P}(\vec{w})$ are given by

$$ea(p) = t_{eh(p)^\vee}\varphi(p), \quad eh(p) \in P^\vee, \varphi(p) \in W.$$

Let $\lambda^\vee \in (P^\vee)^+$ and let $\vec{\lambda}^\vee$ be a minimal length walk to the $\lambda^\vee$-hexagon. Let $W^{\lambda^\vee}$ be a set of minimal coset representatives of $W/W_{\lambda^\vee}$, where $W_{\lambda^\vee}$ is the stabilizer of $\lambda^\vee$ in W. Let

$$\mathcal{P}(\vec{\lambda}^\vee) = \bigsqcup_{w\in W^{\lambda^\vee}} \mathcal{P}(w\vec{\lambda}^\vee) \quad \text{and} \quad \mathcal{P}(\vec{\lambda}^\vee)_{\mu^\vee} = \{p \in \mathcal{P}(\lambda^\vee) \mid \mathrm{eh}(p) = \mu^\vee\}.$$

As points of G/K, the 0th label on these paths is a coset representative of an I-coset in K.

Let $\mathcal{B}(\vec{\lambda}^\vee)_{\mu^\vee}$ be the set of (unlabeled) walks that are obtained by removing the labels from the elements of $\mathcal{P}(\vec{\lambda}^\vee)_{\mu^\vee}$. Let

$$\begin{aligned} \epsilon_+(b) &= \text{number of positive crossings in } b, \\ f(b) &= \text{number of folds in } b. \end{aligned}$$

It is natural to define

$$\dim(b) = \epsilon_+(b) + f(b).$$

since the points of G/K corresponding to b form a "cell" isomorphic to $k^{\epsilon_+(b)} \times (k^\times)^{f(b)}$. If $\mathrm{Card}(k) = q$ then each $b \in \mathcal{B}(\vec{\lambda}^\vee)_{\mu^\vee}$ corresponds to

$$\mathrm{Card}_q(b) = q^{\epsilon_+(b)}(q-1)^{f(b)} \quad \text{elements of } \mathcal{P}(\vec{\lambda}^\vee)_{\mu^\vee}.$$

The *MV-cycles of type* λ *and weight* μ are the elements of the set

$$MV(\lambda)_\mu = \{\text{irreducible components of } \overline{U^- t_\mu K \cap K t_\lambda K}\}.$$

Theorem 1.6.

(a) (Gaussent-Littelmann) The map $\begin{array}{ccc} B(\vec{\lambda}^\vee)_{\mu^\vee} & \longrightarrow & MV(\lambda^\vee)_{\mu^\vee} \\ p & \longmapsto & \overline{Z(p)} \end{array}$ *is a bijection.*

(b) (Mirkovic-Vilonen [MV, 4.6]) All irreducible components of $\overline{U^- t_\mu K \cap K t_\lambda K}$ *have dimension* $\langle \lambda - \mu, \rho \rangle$.

(c) (Anderson [An, Prop. 3, p. 579) $MV(\lambda)_\mu$ *is the set of irreducible components of* $\overline{U^- t_\mu K \cap U^+ t_\lambda K}$ *which are contained in* $\overline{K t_\lambda K}$.

Proof. By Macdonald's spherical function formula

$$s_\lambda = P_\lambda\big|_{q^{-1}=0} = \sum_{\mu \in P^\vee} \mathrm{Card}(B(\lambda)_\mu) X^\mu.$$

Thus the paths of maximal dimension in $\hat{B}(-\lambda^\vee)_{-\mu^\vee}$ have dimension $\langle \lambda^\vee - \mu^\vee, \rho \rangle$. □

References

[BD] Y. Billig and M. Dyer, *Decompositions of Bruhat type for the Kac-Moody groups*, Nova J. Algebra Geom. **3** no. 1 (1994), 11–31.

[KP1] V.G. Kac and D.H. Peterson, *Infinite flag varieties and conjugacy theorems*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 1778-1782.

[KP2] V.G. Kac and D.H. Peterson, *Regular functions on certain infinite dimensional groups*, in "Arithmetic and Geometry" , M. Artin and J. Tate, Eds. 141-166. Birkhäuser, Basel, 1983.