Free algebras $\mathcal{P}$ and $\mathcal{F}$

$\mathcal{P}$, the free algebra generated by $p_1, \dots, p_r$, and
$\mathcal{F}$, the free algebra generated by $f_1, \dots, f_r$,
are graded dual via $\langle\, , \rangle : \mathcal{P} \times \mathcal{F} \to \mathbb{C}$ given by

$$\langle p_{i_1} \cdots p_{i_d}, f_{j_1} \cdots f_{j_\ell} \rangle = \delta_{d\ell}\, \delta_{i_1 j_1} \cdots \delta_{i_\ell j_\ell}$$

and

$$\deg(p_{i_1} \cdots p_{i_d}) = \deg(f_{i_1} \cdots f_{i_d}) = \alpha_{i_1} + \cdots + \alpha_{i_d}$$

is in

$$Q^+ = \mathbb{Z}_{\geq 0}\text{-span}\{\alpha_1, \dots, \alpha_r\}$$

$\mathcal{P}$ has basis $\{p_{i_1}^{(n_1)} \cdots p_{i_m}^{(n_m)}\}$

$\mathcal{F}$ has basis $\{[n_1]! \cdots [n_m]!\, f_{i_1}^{n_1} \cdots f_{i_m}^{n_m}\}$

<u>Dual maps:</u>

$m : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$	$\Delta : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$
$r : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$	$\circ : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$
$^- : \mathcal{P} \to \mathcal{P}$	$^- : \mathcal{F} \to \mathcal{F}$
$m_{an} : \mathcal{P} \to \mathcal{P}$	$\Delta_{an} : \mathcal{F} \to \mathcal{F}$

This means

$$\langle x, y \circ y' \rangle = \langle r(x), y \otimes y' \rangle \quad \text{and} \quad \langle x x', y \rangle = \langle x \otimes x', \Delta(y) \rangle$$

Bialgebra structure on P

$$m : P \otimes P \longrightarrow P \quad \text{given by} \quad m(u \otimes v) = uv,$$

$$r : P \longrightarrow P \otimes P \text{ given by } r(p_i) = p_i \otimes 1 + 1 \otimes p_i;$$

where the product on $P \otimes P$ is given by

$$(x \otimes y)(z \otimes w) = q^{-\langle deg(y), deg(z)\rangle} xz \otimes yw$$

and

$$\bar{\ } : P \longrightarrow P \quad \text{by} \quad \bar{p_i} = p_i \text{ and } \bar{q} = q^{-1}.$$

and

$$I_{p_i^{(n)}} : P \longrightarrow P \quad \text{by} \quad I_{p_i^{(n)}}(x) = x p_i^{(n)}.$$

Bialgebra structure on F

$\circ : F \otimes F \longrightarrow F$ is given by

$$u \circ v = \sum_{\sigma \in S_{k+l}/S_k \times S_l} q^{wt(\sigma, u, v)} \sigma(uv),$$

where $k = l(u)$, $l = l(v)$, the sum is over minimal length coset representatives of $S_k \times S_l$-cosets in S_{k+l},

$$wt(\sigma, u, v) = \sum_{1 \le i < j \le k+l} -\langle u_i, v_{k-j}\rangle, \quad \text{where}$$

u_i is the i^{th} letter in u and

v_{k-j} is the $(k-j)^{th}$ letter in v.

$\Delta: \mathcal{F} \to \mathcal{F} \otimes \mathcal{F}$ is given by

$$\Delta(u) = \sum_{\substack{v,w \in \mathcal{F} \\ vw = u}} v \otimes w$$

and is a homomorphism with respect to the product

$$(x \otimes y) \circ (z \otimes w) = q^{-\langle \deg(y), \deg(z) \rangle} (x \circ z) \otimes (y \circ w).$$

Define

$$\bar{\ }: \mathcal{F} \to \mathcal{F} \quad \text{by} \quad \bar{f_i} = f_i \quad \text{and} \quad \bar{q} = q^{-1}$$

and

$$\Delta_{a^n}: \mathcal{F} \to \mathcal{F} \quad \text{by}$$

$$\Delta_{a^n}(x) = \sum_{\substack{u \in x \\ u = wa^n}} w,$$

so that the sum is over terms in x that end in a^n.

Hence

$$\Delta_{a^n}(f_{j_1} \cdots f_{j_d}) = \begin{cases} f_{j_1} \cdots f_{j_{d-n}}, & \text{if } j_{d-n+1} = \cdots = j_d = a, \\ 0, & \text{otherwise} \end{cases}$$

<u>Key point</u>

$$U^- \hookrightarrow \mathcal{F} \quad \text{and} \quad \mathcal{P} \twoheadrightarrow U^-$$

<u>HW</u>: Make precise: "The Serre relations are the radical of $\langle\, ,\, \rangle$".
and $\langle\, ,\, \rangle: U^- \times U^- \to \mathbb{C}$ is nondegenerate symmetric bilinear form

The quantum group ~~and categorification~~ $\mathcal{U}^-$

Let $\mathring{\mathcal{F}}$ be the vector space $\mathcal{F}$ with product $\circ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$. Define a homomorphism

$$\Phi : \mathcal{F} \longrightarrow \mathring{\mathcal{F}} \qquad \Phi(uv) = u \circ v,$$
$$f_i \longmapsto f_i,$$

so that

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\Phi"] \arrow[dr] & & \mathring{\mathcal{F}} \\ & \mathcal{U}^- \arrow[ur, hook] & \end{tikzcd} \qquad \text{where } \mathcal{U}^- = \mathcal{F}/\ker \Phi$$

As pointed out to me by M. Aguiar, if the Cartan matrix is the 0 matrix then

$\mathcal{F} = T(V) =$ tensor algebra,

$\mathring{\mathcal{F}} = T^\vee(V) =$ suffle algebra,

and

$$\begin{tikzcd} T(V) \arrow[rr, "\varphi"] \arrow[dr] & & T^\vee(V) \\ & S(V) \arrow[ur, hook] & \end{tikzcd} \qquad \text{by}$$

$$\varphi(u_1 \cdots u_\ell) = \sum_{\sigma \in S_\ell} u_{\sigma(1)} \cdots u_{\sigma(\ell)},$$

and φ is the unique map such that

$$\varphi(uv) = u \circ v \quad \text{and} \quad \varphi(f_i) = f_i.$$

Note that φ is an Hopf algebra map.