Cartan matrices, Dynkin diagrams and Quiver data

A symmetrizable Cartan matrix is

$A = (a_{ij})$ with $a_{ii} = 2$,

$a_{ij} \in \mathbb{Z}_{\leq 0}$ and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, for $i \neq j$,

such that there exists a diagonal matrix

$D = \mathrm{diag}(d_1, \ldots, d_r)$ with $d_i = \dfrac{\langle \alpha_i, \alpha_i \rangle}{2}$

and

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

so that DA is the matrix of a symmetric bilinear form

$\langle \, , \, \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ with respect to a basis $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{h}^*$.

Define, following Rouquier [RO, ???],

$$Q_{ij}(u, v) = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } \langle \alpha_i, \alpha_j \rangle = 0, \\ -u^{-\langle \alpha_i^\vee, \alpha_j \rangle} + v^{-\langle \alpha_j^\vee, \alpha_i \rangle}, & \text{if } \langle \alpha_i, \alpha_j \rangle < 0 \text{ and } i \to j, \\ u^{-\langle \alpha_i^\vee, \alpha_j \rangle} - v^{-\langle \alpha_j^\vee, \alpha_i \rangle}, & \text{if } \langle \alpha_i, \alpha_j \rangle < 0 \text{ and } i \leftarrow j. \end{cases}$$

Examples of Cartan matrices

A_r: $\overset{}{\underset{1}{\circ}} - \underset{2}{\circ} - \circ - \cdots - \underset{r}{\circ}$ has $A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$

with $\mathfrak{h}^* = \{\lambda_1 \varepsilon_1 + \cdots + \lambda_{r+1}\varepsilon_{r+1} \mid \lambda_1 + \cdots + \lambda_{r+1} = 0\}$

and $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$.

$$Q = \begin{pmatrix} 0 & v-u & 1 & 1 & \cdots & 1 \\ u-v & 0 & v-u & 1 & \cdots & 1 \\ 1 & u-v & 0 & v-u & \cdot & 1 \\ 1 & 1 & u-v & 0 & \ddots & \vdots \\ \vdots & & & \ddots & 0 & v-u \\ 1 & \cdots & & 1 & u-v & 0 \end{pmatrix}$$ for $\underset{1}{\circ} \to \underset{2}{\circ} \to \circ \to \circ - \cdots \to \underset{n}{\circ}$

A_2: $\circ - \circ$ $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $Q(u,v) = \begin{pmatrix} 0 & v-u \\ u-v & 0 \end{pmatrix}$

$$Q = \begin{array}{c c} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \begin{pmatrix} 0 & v-u \\ u-v & 0 \end{pmatrix} \end{array}$$ for $\overset{+}{\circ} \longrightarrow \overset{-}{\circ}$

$$Q = \begin{array}{c c} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \begin{pmatrix} 0 & u-v \\ v-u & 0 \end{pmatrix} \end{array}$$ for $\overset{+}{\circ} \longleftarrow \overset{-}{\circ}$

B_n:  has

$$A = \begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix}$$

so that

$$DA = \begin{pmatrix} 2 & -2 & & & & \\ -2 & 4 & -2 & & & \\ & -2 & 4 & -2 & & \\ & & -2 & 4 & -2 & \\ & & & & \ddots & \\ & & & & 4 & -2 \\ & & & & -2 & 4 \end{pmatrix}$$

is the matrix of

$\langle\ ,\ \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ on $\mathfrak{h}^* = \mathrm{span}\{\varepsilon_1, \dots, \varepsilon_n\}$ ~~given by~~ given by

$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ in the basis $\{\alpha_1, \dots, \alpha_n\}$ where

$$\alpha_1 = \varepsilon_1, \quad \alpha_2 = \varepsilon_2 - \varepsilon_1, \quad \alpha_3 = \varepsilon_3 - \varepsilon_2, \quad \dots, \quad \alpha_n = \varepsilon_n - \varepsilon_{n-1}.$$

Then

$$Q = \begin{pmatrix} 0 & v-u^2 & 1 & 1 & \cdots & 1 \\ u-v^2 & 0 & v-u & 1 & & \vdots \\ 1 & u-v & 0 & v-u & & \vdots \\ 1 & 1 & u-v & 0 & & \vdots \\ \vdots & \vdots & & & \ddots & 1 \\ \vdots & & & & 0 & v-u \\ 1 & 1 & \cdots & 1 & u-v & 0 \end{pmatrix}$$

for 

$\underline{C_n}$: $\underset{1}{\circ} \Leftarrow \underset{2}{\circ} - \circ - \cdots - \underset{n}{\circ}$ has

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

so that

$$DA = \begin{pmatrix} 4 & -2 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & \ddots & \\ & & & & \ddots & -1 \\ & & & & -1 & 2 \end{pmatrix} \quad \text{is the matrix of}$$

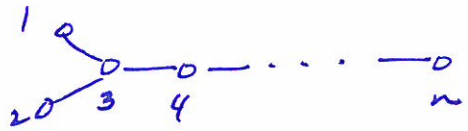
$\langle\, , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \longrightarrow \mathbb{C}$ on $\mathfrak{h}^* = \operatorname{span}\{\varepsilon_1, \dots, \varepsilon_n\}$ given by

$\langle \varepsilon_i ; \varepsilon_j \rangle = \delta_{ij}$ in the basis $\{\alpha_1, \dots, \alpha_n\}$ where

$\alpha_1 = 2\varepsilon_1$, $\alpha_2 = \varepsilon_2 - \varepsilon_1$, $\alpha_3 = \varepsilon_3 - \varepsilon_2, \dots, \alpha_n = \varepsilon_n - \varepsilon_{n-1}$

Then

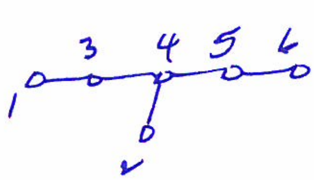
$$Q = \begin{pmatrix} 0 & v^2-u & 1 & 1 & \cdots & \cdots & 1 \\ u^2-v & 0 & v-u & 1 & & & \vdots \\ 1 & u-v & 0 & v-u & & & \vdots \\ 1 & 1 & u-v & \ddots & & & \vdots \\ \vdots & & & & \ddots & & 1 \\ \vdots & & & & & 0 & v-u \\ 1 & \cdots & \cdots & \cdots & 1 & u-v & 0 \end{pmatrix} \quad \text{for } \circ \Leftarrow \circ - \cdots - \circ$$

D_n:  has $D = id$ and

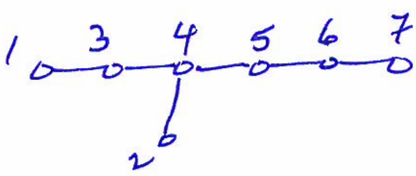
$$A = DA = \begin{pmatrix} 2 & 0 & -1 & & & & \\ 0 & 2 & -1 & & & & \\ -1 & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & \ddots & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}$$ and

$$Q = \Bigg($$

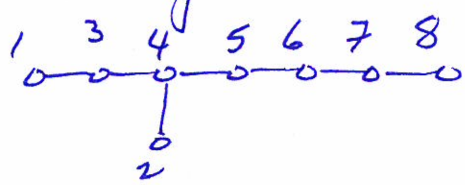
For E_6, E_7, E_8 use the Bourbaki numbering



E_6



E_7



E_8

$\underline{F_4}$:  has

$$A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -2 & 2 & -1 \\ & & -1 & 2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so that

$$DA = \begin{pmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

$\underline{G_2}$ $\quad A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ has $D = \begin{pmatrix} 3 & \\ & 1 \end{pmatrix}$

so that $\quad DA = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$