General partial fractions decomposition

(A)

$$Q = (x-\lambda_1)^{\nu_1} \cdots (x-\lambda_r)^{\nu_r}.$$

$$Q_i = \frac{Q}{(x-\lambda_i)^{\nu_i}}$$

Then

$$\frac{P}{Q} = \frac{A_1}{(x-\lambda_1)^{\nu_1}} + \frac{A_2}{(x-\lambda_2)^{\nu_2}} + \cdots + \frac{A_r}{(x-\lambda_r)^{\nu_r}}$$

with

$$A_i(x) = \sum_{k=0}^{\nu_i-1} \frac{1}{k!} \frac{d^k}{dx^k}\left(\frac{P}{Q_i}\right)\Bigg|_{x=\lambda_i} (x-\lambda_i)^k.$$

In particular:

$$\frac{1}{m} = \frac{A_1}{(x-\lambda_1)^{\nu_1}} + \cdots + \frac{A_r}{(x-\lambda_r)^{\nu_r}}.$$

with
$$A_i = \sum_{k=0}^{\nu_i-1} \frac{1}{k!} \frac{d^k}{dx^k}\left(\frac{1}{Q_i}\right)\Bigg|_{x=\lambda_i} (x-\lambda_i)^k$$

$$= \sum_{k=0}^{\nu_i-1} \frac{1}{k!} \frac{d^k}{dx^k}\left(\frac{(x-\lambda_i)^{\nu_i}}{Q}\right)\Bigg|_{x=\lambda_i} (x-\lambda_i)^k.$$

So

$$1 = \frac{A_1 m}{(x-\lambda_1)^{\nu_1}} + \cdots + \frac{A_r m}{(x-\lambda_r)^{\nu_r}}.$$

and this provides a decomposition into block diagonal form.

Even better

$$1 = \left(a_{10} + a_{11}(x-\lambda_1) + \cdots + a_{1,\nu_1-1}(x-\lambda_1)^{\nu_1-1}\right)\frac{m}{(x-\lambda_1)^{\nu_1}} + \cdots$$

$$\cdots + \left(a_{r0} + a_{r1}(x-\lambda_r) + \cdots + a_{r,\nu_r-1}(x-\lambda_r)^{\nu_r-1}\right)\frac{m}{(x-\lambda_r)^{\nu_1}}$$

$$E_1 x = \left(\begin{array}{c|c} \begin{matrix}\lambda_1 & 1 & \\ & \ddots & 1 \\ & & \lambda_1\end{matrix} & \\ \hline & 0 \end{array}\right) \qquad E_1(x-\lambda_1) = \left(\begin{array}{c|c} \begin{matrix}0 & 1 & \\ & \ddots & 1 \\ & & 0\end{matrix} & \\ \hline & 0 \end{array}\right)$$

and $E_1(x-\lambda_1)^{\nu_1-1} = \left(\begin{array}{c|c} \begin{matrix}0 & \cdots & 1 \\ & & 0\end{matrix} & \\ \hline & 0 \end{array}\right)$ is my top vector

$$\begin{pmatrix}\lambda_1 & 1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda\end{pmatrix}\begin{pmatrix}1 \\ 0 \\ \vdots \\ 0\end{pmatrix} = \lambda_1\begin{pmatrix}1 \\ 0 \\ \vdots \\ 0\end{pmatrix} \qquad \begin{pmatrix}\lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_1 & 1 \\ & & & \lambda_1\end{pmatrix}\begin{pmatrix}0 \\ \vdots \\ 0 \\ 1\end{pmatrix} = \begin{pmatrix}0 \\ \vdots \\ 0 \\ 1 \\ \lambda_1\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0\end{pmatrix}\begin{pmatrix}0 \\ \vdots \\ 0 \\ 1\end{pmatrix} = \begin{pmatrix}\vdots \\ 1 \\ 0\end{pmatrix}$$

Let v be s.t. $E_1(x-\lambda_1)^{\nu_r-1}v \neq 0$.

Partial fractions

(a) Let $\frac{f}{g} \in \mathbb{C}(x)$.

Then there exist $p \in \mathbb{C}[x]$ and $c_{a,i} \in \mathbb{C}$ such that

$$\frac{f}{g} = p + \left(\sum_{a,i} \frac{c_{a,i}}{(x-a)^i} \right)$$

(b) The elements $\left\{ \frac{1}{(x-a)^i} \,\middle|\, i \in \mathbb{Z}_{>0},\ a \in \mathbb{C} \right\} \cup \{ x^i \mid i \in \mathbb{Z}_{\geq 0} \}$.

are a basis of $\mathbb{C}(x)$.

Partial fractions for $\mathbb{Z}$

Let $n = n_1 n_2 \cdots n_k$ with ~~$n_1, \ldots, n_k$ coprime~~, $n_i = p_i^{e_i}$, p_i prime, $e_i \in \mathbb{Z}_{>0}$.

If $m \in \mathbb{Z}$ then there exist $m_1, \ldots m_k$ s.t.

$$\frac{m}{n} = \frac{m_1}{n_1} + \cdots + \frac{m_k}{n_k}$$

$$\frac{m_1}{n_1} + \cdots + \frac{m_k}{n_k} = \frac{nm_1}{n_1} + \cdots + \frac{nm_k}{n_k} = m$$

$$= \left(\frac{n}{n_1}\right) m_1 + \cdots + \left(\frac{n}{n_k}\right) m_k$$

There exist $p_1, \ldots, p_k \in \mathbb{C}[t]$ s.t.

$$\left(\frac{n}{n_1}\right) p_1 + \cdots + \left(\frac{n}{n_k}\right) p_k = 1$$

So

$$\left(\frac{n}{n_1}\right) p_1 m + \cdots + \left(\frac{n}{n_k}\right) p_k m = m \ \checkmark.$$