Tori $\mathbb{C}^g/\mathbb{Z}^g$

Let $V = \mathbb{C}^g = \{(z_1, z_2, \dots, z_g) \mid z_i \in \mathbb{C}\}$

Let $a_1, \dots, a_{2g} \in \mathbb{C}^g$ and

$$\Lambda = \mathbb{Z}\text{ span}\{a_1, a_2, \dots, a_{2g}\}$$

~~Let~~ The corresponding period matrix is

$$\Omega = \begin{pmatrix} \text{---} & a_1 & \text{---} \\ \text{---} & a_2 & \text{---} \\ & \vdots & \\ \text{---} & a_{2g} & \text{---} \end{pmatrix}$$

Let

$$T = \mathbb{C}^g/\Omega = V/\Lambda$$

There is a bijection

$$\left\{ \begin{array}{c} g \text{ dimensional} \\ \text{tori } V/\Lambda \end{array} \right\} \longleftrightarrow GL_{2g}(\mathbb{Z}) \backslash M_{2g \times g}(\mathbb{C}) / GL_g(\mathbb{C})$$

$$\mathbb{C}^g/\Omega \longleftarrow\!\!\shortmid\; \Omega = \begin{pmatrix} \tau \\ \hline 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Coset reps in $GL_{2g}(\mathbb{Z}) \backslash M_{2g \times g}(\mathbb{C}) / GL_g(\mathbb{C})$ are

$$\begin{pmatrix} \tau \\ \hline 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \text{ with } \tau \in \mathcal{T}/GL_{2g}(\mathbb{Z})$$

where $\mathcal{T} = \{\tau \in M_{g \times g}(\mathbb{C}) \mid \det(\operatorname{Im} \tau) \neq 0\}$.

with $GL_{2g}(\mathbb{Z})$ acting on $\mathcal{T}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau = (A\tau + B)(C\tau + D)^{-1}.$$

Abelian varieties

An *abelian variety* is $\mathbb{C}^g/\Omega$ which can be embedded in projective space $\mathbb{CP}^n$.

$$\begin{array}{ccc} \mathbb{C}^g \times \mathbb{C} \simeq q^*L & \longrightarrow & L \simeq \mathbb{C}^g \times_\Lambda \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}^g & \xrightarrow{q} & \mathbb{C}^g/\Omega \end{array}$$

If $\Lambda = \mathbb{Z}\text{-span}\{a_1, a_2, \ldots, a_{2g}\}$ then $\exists$

Λ acts on $\mathbb{C}^g \times \mathbb{C}$ by

$$\lambda \circ (z, \zeta) = (z + \lambda, j(\lambda, z)\zeta)$$

and

$$(\lambda' + \lambda) \circ (z, \zeta) = (z + \lambda + \lambda', j(\lambda', z+\lambda)\, j(\lambda, z)\zeta)$$

$$= (z + \lambda + \lambda', j(\lambda' + \lambda, z)\zeta)$$

so that

$$j(\lambda' + \lambda, z) = j(\lambda', z + \lambda)\, j(\lambda, z).$$

Define a Hermitian form

$H: \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ and $\chi: \Lambda \to U_1(\mathbb{C})$ by

$$j(\lambda, z) = \chi(\lambda) e^{\pi H(z, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)}$$

and

(*) $\quad \operatorname{im} H(\lambda, \lambda') \in \mathbb{Z}$ and $\chi(\lambda + \lambda') = \chi(\lambda)\chi(\lambda') e^{i\pi \operatorname{im} H(\lambda, \lambda')}$

There is a bijection

$$H^1(T, \mathcal{O}_T^*) \equiv \left\{\begin{array}{l}\text{line bundles } L \\ \text{on } \mathbb{C}^g/\Omega\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{pairs } (H, \chi) \text{ which} \\ \text{satisfy } (*)\end{array}\right\}$$

$\{$ample line bundles$\ldots\} \longleftrightarrow \{\ldots (H, \chi)$ with

Let
$$\Delta = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_g \end{pmatrix} \quad \text{with } d_1 | d_2 | \cdots | d_g$$

The <u>Siegel upper half space</u> of degree g is
$$\mathcal{G}_g = \{ \tau \in M_{g \times g}(\mathbb{C}) \mid \tau^t = \tau,\ \operatorname{Im} \tau > 0 \}$$

Let
$$Sp(\Delta, \mathbb{Z}) = \left\{ M \in GL_{2g}(\mathbb{Z}) \;\middle|\; M \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \right\}$$

$$Sp(\Delta, \mathbb{R}) = \left\{ M \in GL_{2g}(\mathbb{R}) \;\middle|\; M \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \right\}.$$

There is a bijection

$$Sp(\Delta, \mathbb{Z}) \backslash \mathcal{G}_g \longleftrightarrow \left\{ \begin{array}{c} \text{polarized abelian varieties} \\ \text{of type } \Delta \end{array} \right\}$$

An <u>principally polarized abelian variety</u> is a polarized abelian variety of type $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

The <u>theta function</u> $\Theta : \mathcal{G}_g \times \mathbb{C}^g \to \mathbb{C}$
$$\Theta(\tau, z) = \sum_{l \in \mathbb{Z}^g} e^{2\pi i \left(\frac{1}{2} l \tau l^t + l z^t \right)}$$
is a section of $L(H, 1)$ ~~for~~ $\mathbb{C}^g \Big/ \begin{pmatrix} \tau \\ id \end{pmatrix}$

The Heisenberg group

Simplest version: $G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

The version in Mumford Theta I

$G' = \mathcal{U}_1(\mathbb{C}) \times \mathbb{R} \times \mathbb{R}$ with

$(\lambda, a, b)(\lambda', a', b') = (\lambda \lambda' e^{2\pi i b a'}, a + a', b + b')$

Let $\Gamma = \left\{ \begin{pmatrix} 1 & 0 & n \\ & 1 & 0 \\ & & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\} \subseteq Z(G) \leq G.$

Then $G/\Gamma \simeq \mathcal{U}_1(\mathbb{C}) \times \mathbb{R} \times \mathbb{R}$, where $\begin{aligned} \mathbb{R}/\mathbb{Z} &\xrightarrow{\sim} \mathcal{U}_1(\mathbb{C}) \\ z &\longmapsto e^{2\pi i z} = \lambda \end{aligned}$

The version in Kac-Peterson

$$N_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \times \mathbb{R} = \{(\alpha, \beta, t) \mid \alpha \in \mathfrak{h}_{\mathbb{R}}^*, \beta \in \mathfrak{h}_{\mathbb{R}}^*, t \in \mathbb{R}\}$$
$$\simeq \mathbb{R}^{\ell} \times \mathbb{R}^{\ell} \times \mathbb{R} \simeq \mathbb{C}^{\ell} \times \mathbb{R}$$

with $(\alpha, \beta, t)(\alpha', \beta', t') = (\alpha + \alpha', \beta + \beta', t + t' + \tfrac{1}{2}(\langle \alpha', \beta \rangle - \langle \alpha, \beta' \rangle))$

~~The group~~

The version "in Cherednik"