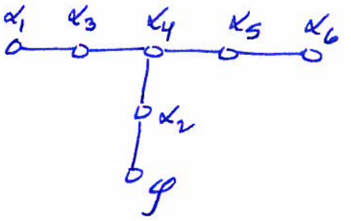
Computation of the crystal of $L(\omega_1)$ for E_6 to volkasthompsongeorge 26.04.2010. ①

E_6 data from Bourbaki Plate V

$$\mathfrak{h}^*_{\mathbb{R}} = \{ (\xi_1, \dots, \xi_8) \in \mathbb{R}^8 \mid \xi_6 = \xi_7 = -\xi_8 \}$$



$$R = \left\{ \begin{array}{l} \pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 5, \\ \pm \frac{1}{2}\left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} \varepsilon_i\right) \mid \sum_{i=1}^{5} \nu(i) \text{ is even} \end{array} \right\}$$

where $\varepsilon_1, \dots, \varepsilon_8$ is an orthonormal basis of $\mathbb{R}^8$.

Card $(R) = 72$

$$\alpha_1 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_8) - \tfrac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$$

$$\alpha_2 = \varepsilon_1 + \varepsilon_2$$

$$\alpha_3 = \varepsilon_2 - \varepsilon_1$$

$$\alpha_4 = \varepsilon_3 - \varepsilon_2$$

$$\alpha_5 = \varepsilon_4 - \varepsilon_3$$

$$\alpha_6 = \varepsilon_5 - \varepsilon_4$$

and
$$\varphi = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$$
$$= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2$$

$P/Q \simeq \mathbb{Z}/3\mathbb{Z}$ and $h = 12$ and $|W_0| = 2^7 \cdot 3^4 \cdot 5$.

The fundamental weights are

$$\omega_1 = \tfrac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6)$$
$$= \tfrac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6)$$

$$\omega_2 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8)$$
$$= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

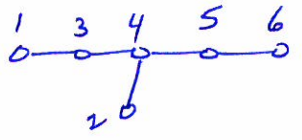
$$\omega_3 = \tfrac{5}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \tfrac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)$$
$$= \tfrac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6)$$

$$\omega_4 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8$$
$$= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6$$

$$\omega_5 = \tfrac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_4 + \varepsilon_5$$
$$= \tfrac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6)$$

$$\omega_6 = \tfrac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5$$
$$= \tfrac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)$$

The Cartan matrix for



is

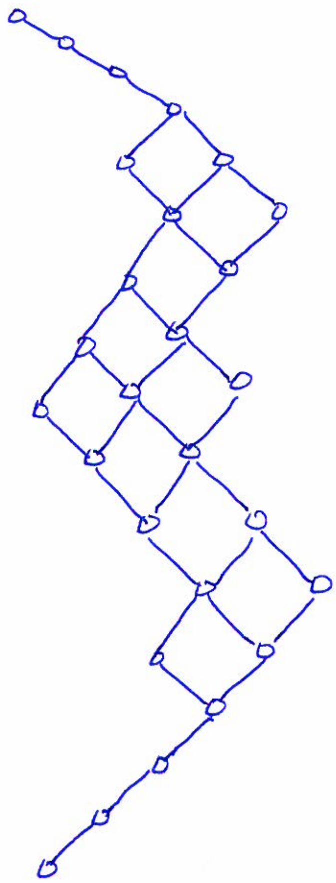
$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Let $(a_1, a_2, a_3, a_4, a_5, a_6) = \sum_{i=1}^{n} a_i \omega_i$

so that the simple roots $\alpha_j = \sum_{i=1}^{n} \langle \alpha_j, \alpha_i^\vee \rangle \omega_i$ correspond to the rows of the Cartan matrix.

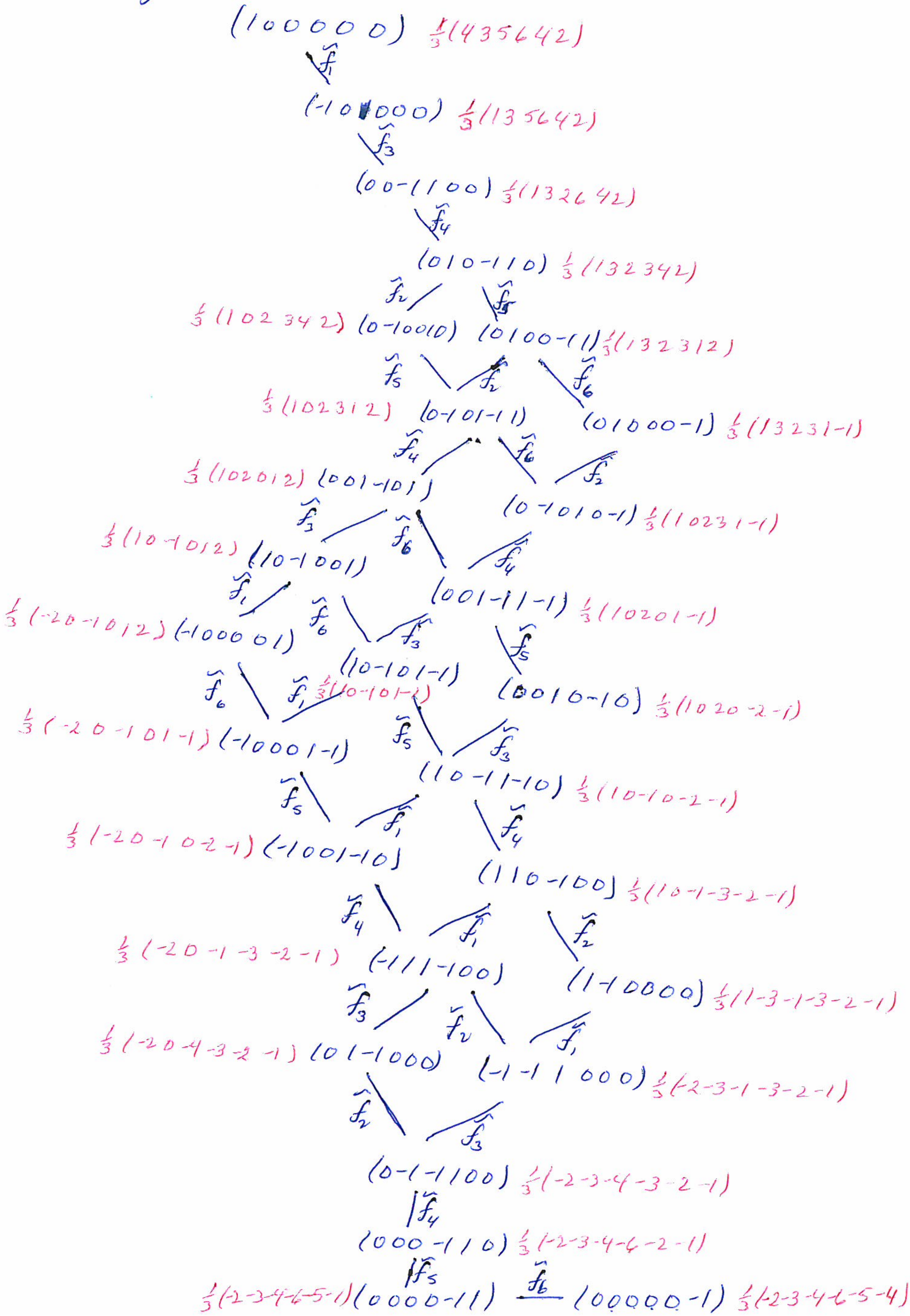
Then $\omega_1 = (100000)$ and the crystal generated by the straight line path to ω_1 is $B(\omega_1)$ given by the figure on the following page.

Note that the poset of this crystal is

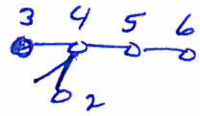


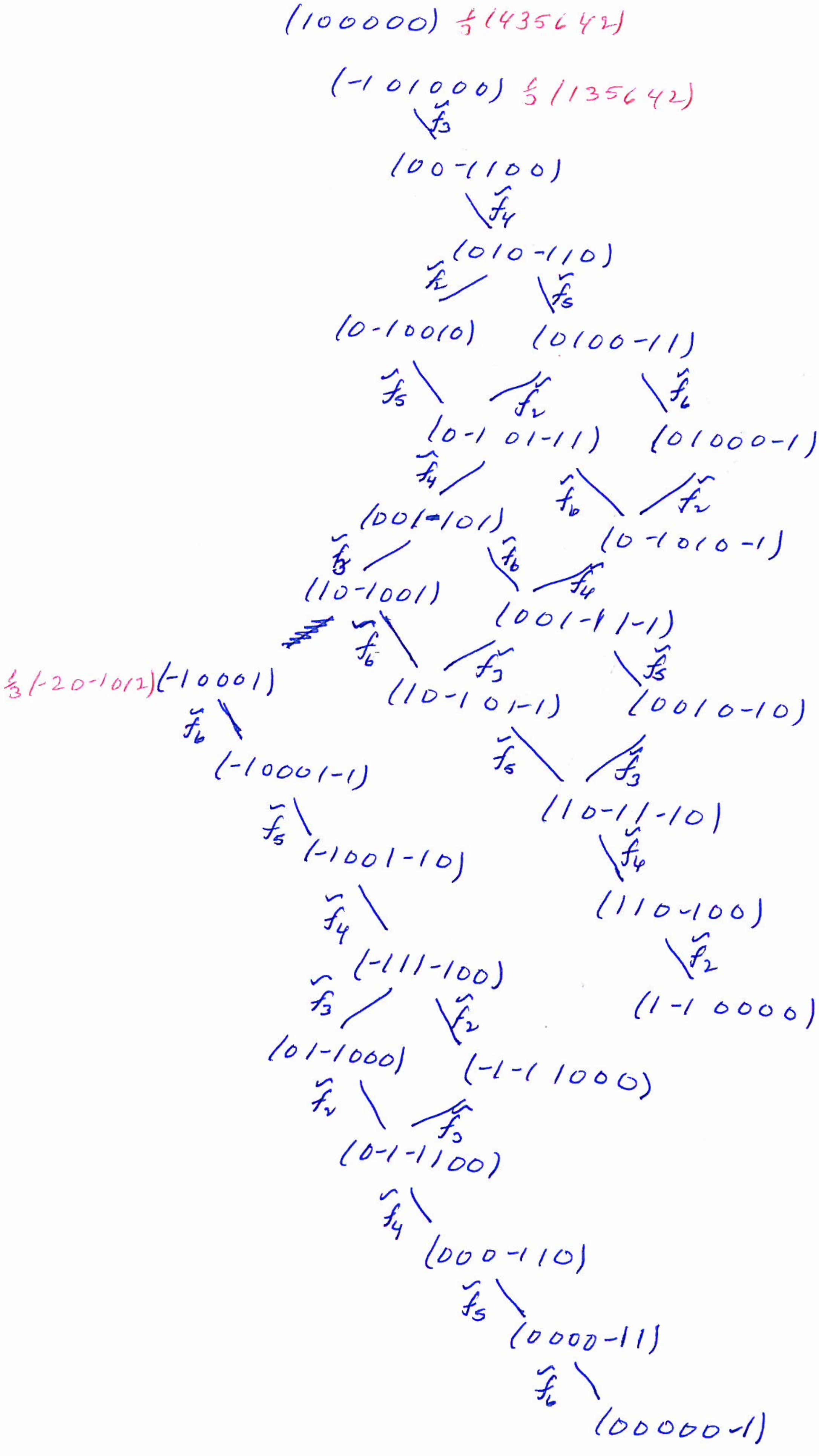
which is Figure 3 on p 489 of J. Stembridge, On minuscule representations, plane partitions, and involutions in complex Lie groups, Duke Math. J. $\underline{\underline{73}}$ No 2 (1994) 469–490.

So the crystal is



The same crystal restricted to $SO(10) \times U(1)$





The same crystal restricted to $SU(5)\times U(1)^2$ 3 4 5 6

⑥

$\frac14(00000)$ (100000) $\frac13(435642)$

$\frac14(35\,\cancel{6}\,42)$ (-101000) $\frac13(135642)$

$\tilde f_3$

$(00-1100)$

$\tilde f_4$

$(010-110)$

$\tilde f_5$

$\frac14(-1,1,5,4,2)$ $\frac13(102342)$ $(0-10010)$ $(1000-11)$

$\tilde f_5$ $\tilde f_6$

$(0-101-11)$ $(01000-1)$ $\frac13(1323\,1\,-1)$

$\tilde f_4$ $\tilde f_6$

$(001-101)$ $(0-1010-1)$

$\tilde f_3$ $\tilde f_6$ $\tilde f_4$

$(10-1001)$ $(001-11-1)$

$\tilde f_6$ $\tilde f_3$ $\tilde f_5$

$\frac13(-20\,1\,0\,1\,2)$ (-100001) $(10-101-1)$ $(0010-10)$

$\frac14(22444)$

$\tilde f_6$ $\tilde f_5$ $\tilde f_3$

$(-10001-1)$ $(10-11-10)$

$\tilde f_5$ $\tilde f_4$

$(-1001-10)$ $(110-100)$ $\frac13(10-10-2-1)$

$\tilde f_4$

$(-111-100)$ $(1-10000)$ $\frac13(1-3\,-1\,-3-2\,-1)$

$\tilde f_3$ $\frac14(-5-3-6-4-2)$

$\frac13(-20-4-3-2-1)$ $(01-1000)$ $(-1-11000)$ $\frac13(-2\,-3\,-1\,-3-2-1)$

$\tilde f_3$

$(0-1-1100)$

$\tilde f_4$

$(000-110)$

$\tilde f_5$

$(0000-11)$

$\tilde f_6$

$(00000-1)$ $\frac14(-2-2-4-4-4)$

Let $L_\beta = \sum_{i=1}^{6} 3(T_i)_\beta T_i$, for $\beta = 1, 2, \dots, 27$

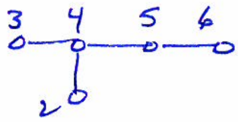
so that L_β has entries $(L_\beta)_\alpha = \sum_{i=1}^{6} 3(T_i)_\beta (T_i)_\alpha$, for $\alpha = 1, \dots 27$

i.e. $L_\beta = \sum_{i=1}^{6} \langle \tau_i, w_\beta \omega_1 \rangle T_i$.

so that L_β acts on $v_{w_\alpha \omega_1}$ by

$$\langle L_\beta, w_\alpha \omega_1 \rangle = \sum_{i=1}^{6} \langle \tau_i, w_\beta \omega_1 \rangle \langle \tau_i, w_\alpha \omega_1 \rangle.$$

For $SO(10)$



$\alpha_2 = \varepsilon_1 + \varepsilon_2$ $\qquad$ $\omega_2 = \frac{1}{2}(\varepsilon_5 + \varepsilon_4 + \varepsilon_3 + \varepsilon_2 + \varepsilon_1)$

$\alpha_3 = \varepsilon_2 - \varepsilon_1$ $\qquad$ $\omega_3 = \frac{1}{2}(\varepsilon_5 + \varepsilon_4 + \varepsilon_3 + \varepsilon_2 - \varepsilon_1)$

$\alpha_4 = \varepsilon_3 - \varepsilon_2$ $\qquad$ $\omega_4 = \varepsilon_5 + \varepsilon_4 + \varepsilon_3$

$\alpha_5 = \varepsilon_4 - \varepsilon_3$ $\qquad$ $\omega_5 = \varepsilon_5 + \varepsilon_4$

$\alpha_6 = \varepsilon_5 - \varepsilon_4$ $\qquad$ $\omega_6 = \varepsilon_5$

Then

$$\omega_2 = \tfrac{1}{2}\alpha_6 + \alpha_5 + \tfrac{3}{2}\alpha_4 + \tfrac{3}{4}\alpha_3 + \tfrac{5}{4}\alpha_2$$
$$\omega_3 = \tfrac{1}{2}\alpha_6 + \alpha_5 + \tfrac{3}{2}\alpha_4 + \tfrac{5}{4}\alpha_3 + \tfrac{3}{4}\alpha_2$$
$$\omega_4 = \alpha_6 + 2\alpha_5 + 3\alpha_4 + \tfrac{3}{2}\alpha_3 + \tfrac{3}{2}\alpha_2$$
$$\omega_5 = \alpha_6 + 2\alpha_5 + 2\alpha_4 + \alpha_3 + \alpha_2$$
$$\omega_6 = \tfrac{1}{2}(2\alpha_6 + 2\alpha_5 + 2\alpha_4 + \alpha_3 + \alpha_2)$$