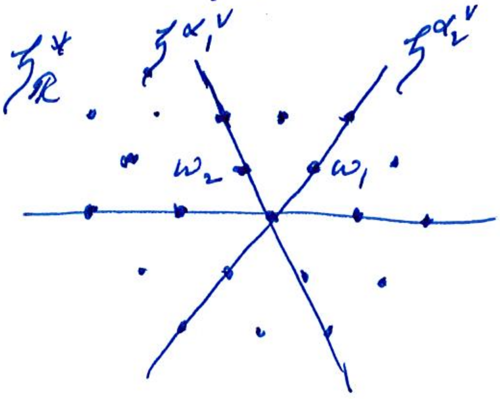
Representation Theory lecture 7, 11.08.2015 ①

Univ. of Melbourne

The initial data - An Example

Type $SL_3(\mathbb{C})$: $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$



$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\text{-span}\{\omega_1, \omega_2\}$
$= \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}$

with $W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$,

the dihedral group of order 6 acting on $\mathfrak{h}^*_{\mathbb{R}}$ (and $\mathfrak{h}^*_{\mathbb{Z}}$) by

s_i is reflection in $\mathfrak{h}^{\alpha_i^\vee}$.

Choose a chamber C.

The positive side of $\mathfrak{h}^{\alpha_i}$ is towards C.

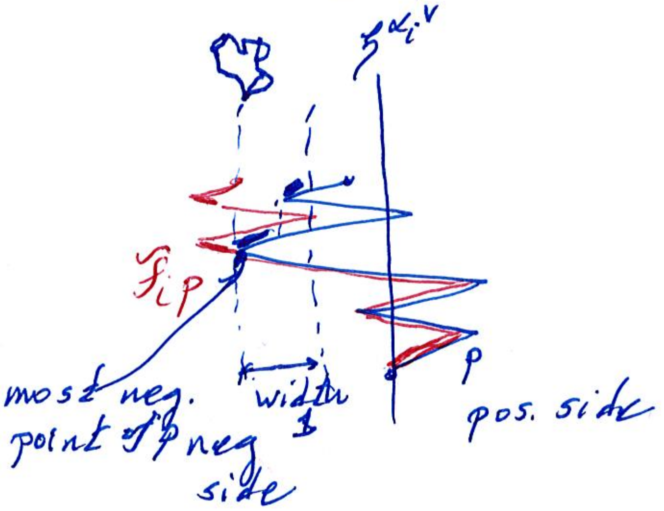
The value $\langle \lambda, \alpha_i^\vee \rangle$ is the distance from λ to $\mathfrak{h}^{\alpha_i^\vee}$

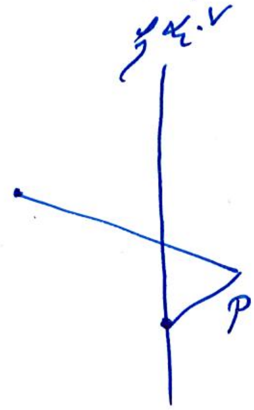
where $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

A path is a piecewise linear function

$p : [0,1] \to \mathfrak{h}^*_{\mathbb{R}}$ with $p(1) \in \mathfrak{h}^*_{\mathbb{Z}}$.

Define <u>root operators</u> $\tilde f_1, \tilde f_2$ on paths by





has $\tilde f_i p$.

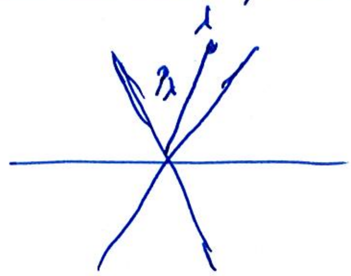
Define <u>root operators</u> $\tilde e_1, \tilde e_2$ on paths by

$$\tilde e_i(p') = \begin{cases} p, & \text{if } p' = \tilde f_i p \\ 0, & \text{otherwise} \end{cases}$$

A <u>crystal</u> is a collection of paths B which is closed under the action of the root operators.

Let $\lambda \in C \cap \mathfrak{h}^*_{\mathbb{Z}}$. The <u>straight line path to λ</u> is

$$p_\lambda : [0,1] \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$$
$$t \longmapsto t\lambda$$



The <u>irreducible</u> <u>crystal</u> $B(\lambda)$ is

$$B(\lambda) = \left\{ \tilde f_{i_1} \cdots \tilde f_{i_\ell} p_\lambda \;\middle|\; \begin{array}{l} i_j \in \{1,2\} \\ \ell \in \mathbb{Z}_{\geq 0} \end{array} \right\}$$

Let B be a crystal.

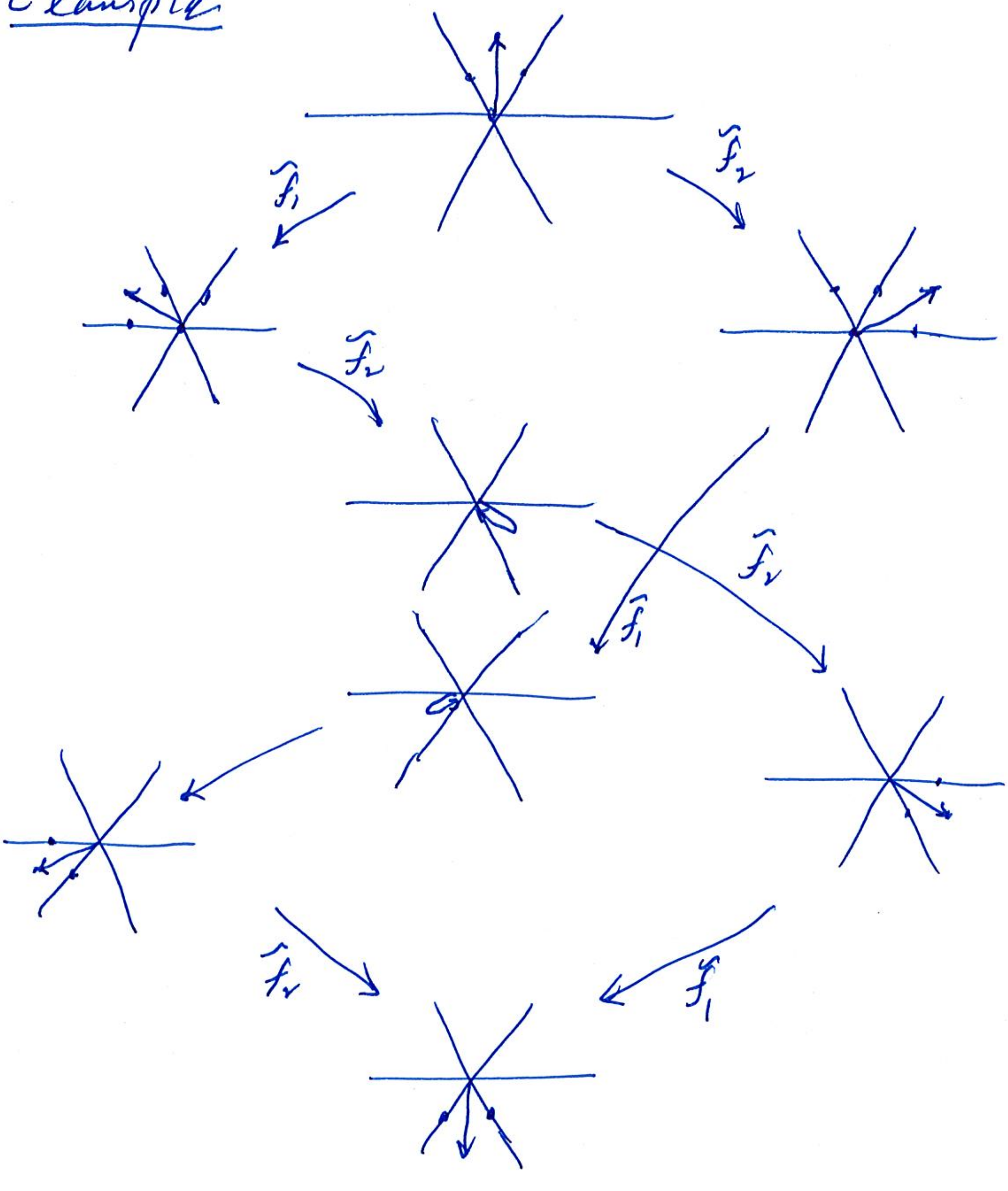
The crystal graph of B is the labeled graph with

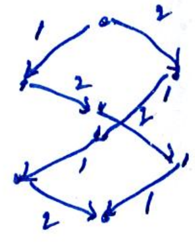
vertices: $p \in B$.

labeled edges: $p' \xleftarrow{i} p$ if $p' = \tilde{f}_i p$

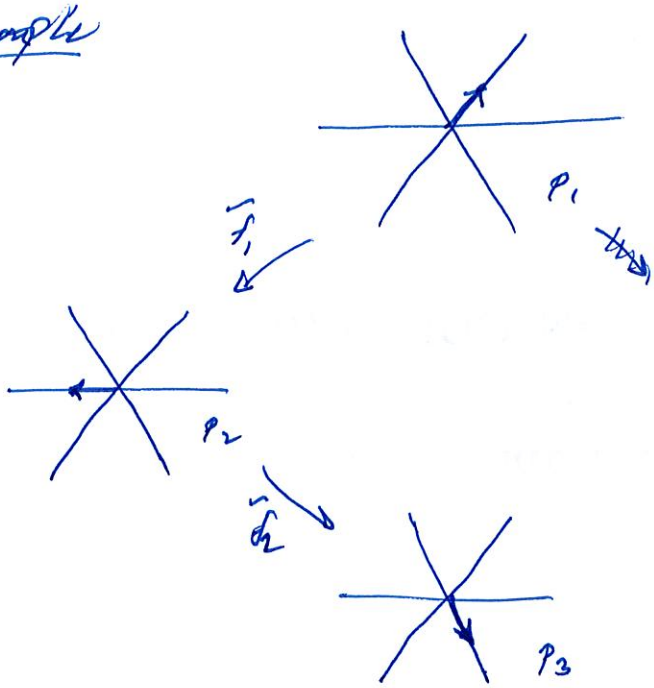
Two crystals B_1 and B_2 are isomorphic if their crystal graphs are isomorphic (as labeled graphs).

Example





Example



Example