Lecture 2: Representation Theory 30 July 2015

Group algebras

Let G be a group.

Example: $G = \{111, X1, 1X, ✱, XX, XX\}$

The group algebra of G is
the vectorspace $\mathbb{C}G$ generated by G with
product determined by the product in G.

Example $(3 \cdot 1X + 7\, ✱)(17\, XX + 12\, X1)$

$= 51 \cdot \frac{1X}{XX} + 36 \cdot \frac{1X}{X1} + 119\, \frac{✱}{XX} + 84\, \frac{✱}{X1}$

$= 51\, X1 + 36\, XX + 119\, 1X + 84\, XX.$

The convolution algebra of functions on G is

$$C(G) = \{f : G \to \mathbb{C}\} \quad \text{with}$$

$$(f_1 * f_2)(g) = \sum_{\substack{h,k \in G \\ hk = g}} f_1(h) f_2(k) = \sum_{h \in G} f_1(h) f_2(gh^{-1})$$

Proposition Let G be a finite group. Then

$$\begin{aligned} C(G) &\longrightarrow \mathbb{C}G \\ f : G \to \mathbb{C} &\longmapsto \sum_{g \in G} f(g)\, g \end{aligned}$$

is an algebra isomorphism.

Other examples of algebras

(1) Polynomials on $x_1, \ldots, x_n$: $\mathbb{C}[x_1, \ldots, x_n]$

Example: $(3x_2 + 4x_1x_3^2 + 2x_1x_2)(5x_2^2x_3 + 6x_1^3 - 12x_1x_3^2)$

$$= 15x_2^3x_3 + 18x_1^3x_2 - 36x_1x_2x_3^2$$
$$+ 20x_1x_2^2x_3^3 + 24x_1^4x_3^2 - 48x_1^2x_3^4$$
$$+ 10x_1x_2^3x_3 + 12x_1^4x_2 - 24x_1^2x_2x_3^2$$

(2) $n \times n$ matrices: $M_n(\mathbb{C})$

Example: $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\left(\begin{pmatrix} 8 & 6 \\ 3 & 8 \end{pmatrix} + 7\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} + 2\begin{pmatrix} 6 & 9 \\ 4 & 3 \end{pmatrix}\right)$

$$= \begin{pmatrix} 25 & 36 \\ 34 & 32 \end{pmatrix} + \begin{pmatrix} 77 & 93 \\ 49 & 7 \end{pmatrix} + \begin{pmatrix} 40 & 54 \\ 56 & 78 \end{pmatrix} = \begin{pmatrix} 132 & 183 \\ 139 & 117 \end{pmatrix}$$

Theorem Let G be a finite group. Then there exists $\ell \in \mathbb{Z}_{>0}$ and $n_1, n_2, \ldots, n_\ell \in \mathbb{Z}_{>0}$ such that

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_\ell}(\mathbb{C})$$

Example

$$M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) = \left\{ \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & a_7 \\ 0 & 0 & a_8 & a_9 & a_{10} \\ 0 & 0 & a_{11} & a_{12} & a_{13} \end{pmatrix} \middle| \; a_i \in \mathbb{C} \right\}$$