Representation Theory 2015 Additional Notes

Let G be a connected complex reductive group.

$\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G.

$\mathfrak{h} = \mathrm{Hom}(\mathbb{C}, \mathfrak{h})$ and $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$

where $\mathfrak{h} = \mathrm{Lie}(T)$, for a maximal torus $T \subseteq G$.

Let M be a finite dimensional ~~$\mathfrak{g}$~~ $\mathfrak{g}$-module.

As $\mathfrak{h}$-modules, $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \left\{ m \in M \;\middle|\; \begin{array}{l} \text{if } h \in \mathfrak{h} \text{ then} \\ \mu(h)m = hm \end{array} \right\}$

A __weight__ of M is $\mu \in \mathfrak{h}^*$ with $M_\mu \neq 0$.

A __root__ of $\mathfrak{g}$ is $\alpha \in \mathfrak{h}^*$ with $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$.

$$\mathfrak{g} = \Big(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\Big) \oplus \mathfrak{h}.$$

Let B be a Borel subgroup with $B \supseteq T$.

$\mathfrak{b} = \mathrm{Lie}(B)$ with $\mathfrak{g} \supseteq \mathfrak{b} \supseteq \mathfrak{h}$.

$\mathfrak{b}$ is an $\mathfrak{h}$-submodule of $\mathfrak{g}$.

A __positive root__ is $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$ such that $\mathfrak{b}_\alpha \neq 0$.

$$\mathfrak{b} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in R^+} \mathfrak{b}_\alpha\Big).$$

$G = GL_n(\mathbb{C})$ has

$$\mathcal{O}_G = \mathbb{C}[t_{ij}, d^{-1}] \quad \text{where } \begin{array}{l} t_{ij} : G \longrightarrow \mathbb{C} \\ (g_{k\ell}) \longmapsto g_{ij}. \end{array}$$

and $d = \det(\cancel{t} \; t_{ij}) = \sum_{w \in S_n} \det(w) \, \cancel{t}_{1w(1)} \cdots t_{nw(n)}$.

The tangent vectors at 1 in G are

$$\xi_X : \mathcal{O}_G \longrightarrow \mathbb{C} \quad \text{given by } \xi_X(f) = f(1 + tX)$$

for $X \in M_n(\mathbb{C})$ and so

$$\mathfrak{g} = \mathfrak{gl}_n = M_n(\mathbb{C})$$

The coproduct is given by $\Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}$

Recall that, when $n = 1$, then $GL_1(\mathbb{C}) = \mathbb{C}^\times$

$$\mathcal{O}_G = \mathbb{C}[t_{ij}, d^{-1}] = \mathbb{C}[t, t^{-1}]$$

"Equivalences" of categories

$$\{\text{Lie groups}\} \xrightarrow{Lie} \{\text{Lie algebras}\}$$

$$G \longmapsto \mathfrak{g} = Lie(G) = T_1(G)$$
$$\varphi \longmapsto d\varphi$$

where

$T_1(G) = \{\text{tangent vectors}_{\xi_1} \text{ to } G \text{ at } 1\}$

$\leftrightarrow \{\text{one parameter subgroups}^{\gamma} \text{ of } G\}$

$\leftrightarrow \{\text{left invariant vector fields}_{\xi} \text{ on } G\}$

with $\xi_1 f = (\xi f)(1)$ and $\gamma_1 = \frac{d}{dt} f(\gamma(t))\big|_{t=0}$

On morphisms, Lie is denoted d:

If $\varphi : G \to H$ then $C^\infty(H) \xrightarrow{\varphi^*} C^\infty(G)$
$$f \longmapsto f \circ \varphi$$

and

$d\varphi : \mathfrak{g} \to \mathfrak{h}$ is given by

$d\varphi(\xi_1) = \xi_1 \circ \varphi^*$, if ξ_1 is a tangent vector to G at 1

$d\varphi(\xi) = \xi \circ \varphi^*$, if ξ is a left invariant vector field,

$d\varphi(\gamma) = \varphi \circ \gamma$, if γ is a one parameter subgroup.

The inverse functor is $\exp : \mathfrak{g} \to G$
$$tX \longmapsto e^{tX}, \text{ where } e^{tX} = \gamma(t).$$

"Equivalences" of categories

Tangent spaces

(1) $\{\text{Lie groups}\} \xrightarrow{\text{Lie}} \{\text{Lie algebras}\}$

$$G \longmapsto \mathfrak{g} = \mathrm{Lie}(G) = T_1(G)$$

$$\varphi: G \to H \longmapsto d\varphi: \mathfrak{g} \to \mathfrak{h}.$$
$$\gamma \mapsto \varphi\circ\gamma$$

The inverse functor is determined by

$$\mathfrak{g} \xrightarrow{\exp} G$$
$$tX \longmapsto e^{tX} = \gamma_X(t),$$

if $\gamma_X: \mathbb{R} \to G$ is the one parameter subgroup corresponding to X.

"Weyl's unitary trick"

(2) $\left\{\begin{matrix}\text{complex reductive}\\ \text{algebraic groups}\end{matrix}\right\} \xrightarrow{K} \left\{\begin{matrix}\text{compact}\\ \text{Lie groups}\end{matrix}\right\} \xrightarrow{\mathcal{O}_G} \{\text{Hopf algebras}\}$

$$G \longrightarrow K(G) \longmapsto \mathcal{O}_G$$

$G \longmapsto K$, the maximal compact subgroup of G

$\varphi: G \to H \longmapsto \mathrm{Res}^G_K(\varphi)$: $K(G) \to K(H)$

The inverse functor is given by

$\mathcal{O}_G$ = algebra of coordinate functions of finite dimensional representations of $K(G)$

and U is recovered from $\mathcal{O}_G$ as $K = \{$~~primitive~~ grouplike elements$\}$ of $\mathcal{O}_G$.

(3) "Differential operators" $\{\text{Lie algebras}\} \xrightarrow{U} \{\text{associative algebras}\}$

$\mathfrak{g} \longmapsto U\mathfrak{g}$, the enveloping algebra of $\mathfrak{g}$.

Here U is the left adjoint to the functor

$$L: \{\text{associative algebras}\} \to \{\text{Lie algebras}\}$$

given by

$L(A) = A$ with bracket $[a_1, a_2] = a_1a_2 - a_2a_1$.

and $\mathfrak{g}$ is recovered from $U\mathfrak{g}$ as

$\mathfrak{g} = \{\text{primitive elements in } U\mathfrak{g}\}$.

"Rootsystems"

(4) $\{\mathbb{Z}\text{-reflection groups}\} \longleftrightarrow \left\{\begin{array}{l}\text{complex reductive}\\ \text{algebraic groups}\end{array}\right\}$

$$\{\mathfrak{h}_{\mathbb{Z}}, W_0\} \longmapsto G$$

where

$$\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}^\times, T) \quad \text{and} \quad \mathfrak{h}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^\times)$$

and $W_0 = N(T)/T$ where $N(T)$ is the normalizer of T in G

The inverse functor is given by letting G be the group generated by

$$x_\alpha(c),\ x_{-\alpha}(c) \text{ and } h_\lambda(t), \quad \text{for } \alpha \in R^+,\ c \in \mathbb{C},\ \lambda \in \mathfrak{h}_{\mathbb{Z}},\ t \in \mathbb{C}^\times$$

with relations

$$x_\alpha(c_1)x_\alpha(c_2) = x_\alpha(c_1+c_2), \ldots$$

$$h_\lambda(t_1)h_\lambda(t_2) = h_\lambda(t_1t_2), \quad h_\lambda(t)h_\mu(t) = h_{\lambda+\mu}(t), \ldots$$

(5) $\{\mathbb{Z}_p\text{-reflection groups}\} \longleftrightarrow \{p\text{-compact groups}\}$

$$(\mathfrak{h}_{\mathbb{Z}_p}, W_0) \longmapsto BG$$