Tutorial Sheet for Lecture 1

MAST90017 Representation Theory
Semester II 2015
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(1) Define category, define the category of algebras, and prove that the category of algebras is a category.

(2) Define category, define the category of vector spaces, and prove that the category of vector spaces is a category.

(3) Define category, define the category of sets, and prove that the category of sets is a category.

(4) Define category, define the category of rings, and prove that the category of rings is a category.

(5) Define category, define the category of fields, and prove that the category of fields is a category. Show that all morphisms in the category of fields are injective.

(6) Let A be an algebra. Define category, define the category of A-modules, and prove that the category of A-modules is a category.

(7) Define tensor product $V \otimes W$ of vector spaces and show that a function $f\colon V \times W \to Z$ is bilinear if and only if $f\colon V \otimes W \to Z$ is a linear transformation.

(8) Let M be a finite dimensional A-module. Let $b_1, \ldots, b_n$ and $v_1, \ldots, v_d$ be two bases of M. Let

$$\rho_b\colon A \to M_d(\mathbb{C}) \qquad \text{and} \qquad \rho_v\colon A \to M_d(\mathbb{C})$$

be the corresponding algebra homomorphisms. Let $a \in A$. Determine a formula for $\rho_v(a)$ in terms of $\rho_b(a)$ and the transition matrix between the two bases.

(9) Let $f\colon M \to N$ be an A-module homomorphism. Show that $\ker f$ is a submodule of M and $\operatorname{im} f$ is a submodule of N.

(10) Let $d \in \mathbb{Z}_{>0}$. Show that $M_d(\mathbb{C})$ is an algebra.

(11) Show that $\mathbb{C}[x_1, \ldots, x_n]$ is an algebra.

(12) Show that $\mathbb{C}[x]$ is an algebra.

(13) Show that $\mathbb{C}$ is an $\mathbb{R}$-algebra.

(14) Let $f\colon A \to B$ be a morphism of algebras. Define a map

$$\begin{array}{rccc} \mathrm{Res}_A^B\colon & \{B\text{-modules}\} & \longrightarrow & \{A\text{-modules}\} \\ & M & \longmapsto & M \end{array} \qquad \text{given by setting} \quad am = f(a)m,$$

for $m \in M$ and $a \in A$. Define functor and show that Res_A^B is a functor.

(15) Let $f\colon A \to B$ be a morphism of algebras and let Res_A^B be as defined in the previous exercise. Show that if f is surjective and M is a simple B-module then $M = \mathrm{Res}_A^B(M)$ is a simple A-module.

(16) Let A be an algebra and let I be an ideal of A. Show that A/I is an algebra and if M is a simple A/I-module then M is a simple A-module.

(17) Show that a simple module is indecomposable and give an example of an indecomposable module that is not simple.