The Language of Mathematics

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1. The grammar of mathematics

- **Definitions** are the foundation of mathematics.
- Theorems are the landmarks of mathematics.
- **Proofs** are the explanation of mathematics.

Learning to read, write and speak mathematics is a skill that anyone can learn. Like all languages, it requires lots of practice to use it fluently.

Like all languages, the grammar of quality mathematical communication is very rigid. The grammar of a definition is:

A <u>noun</u> is a	such	that		
(a) If	then	,	and	
(b) If	then	,	and	
(c) If	then	,	and .	•••

An adjective is most conveniently defined by putting it in the form of a noun:

An <u>adjective</u> noun is a ______, such that (a) If _______ then _____, and (b) If _______ then _____, and ...

Sometimes definitions of adjectives take the form:

Let S be a <u>noun</u>. The <u>noun</u> S is <u>adjective</u> if S satisfies (a) If ______then ____, and (b) If ______then ____, and ...

The words "let" and "assume" are synonyms for "if". The grammar of a lemma, proposition or theorem (or any other statement) is:

If _____ then ____.

Two special constructions in mathematical language are:

There exists such that .

and

There exists a unique _____ such that ____.

It is **impossible** to prove a statement without being able to write down the definitions of all the terms in the statement.

2. How to do proofs

There *is* a certain "formula" or method to doing proofs. Some of the guidelines are given below. The most important factor in learning to do proofs is practice, just as when one is learning a new language.

1. There are very few words needed in the structure of a proof. Organized in rows by synonyms they are:

```
To show
Assume, Let, Suppose, Define, If
Since, Because, By
Then, Thus, So
There exists, There is
Recall, We know, But
```

2. The overall structure of a proof is a block structure like an outline. For example:

3. Any proof or section of proof begins with one of the following:

(a) To show: If <u>A</u> then <u>B</u>.
(b) To show: There exists <u>C</u> such that <u>D</u>.
(c) To show: <u>E</u>.

4. Immediately following this, the next step is

Case (a) Assume the ifs and `to show' the thens. The next lines usually are

• Assume <u>A</u>. • To show: <u>B</u>.

Case (b) To show an object exists you must find it. The next lines usually are

• Define _____.

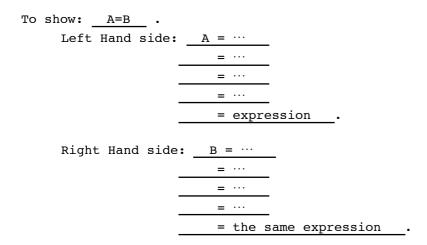
• To show: ______ satisfies ____.

Case (c) Rewrite the statement in <u>E</u> by using a definition. The next line is usually
To show <u>E'</u>.

A useful guideline is, "Don't think too much." Following the "method" usually produces a proof without thinking. Most of doing proofs is simply rewriting what has come just before in a different form by plugging in a definition.

There are some kinds of proofs which have a special structure.

Proofs of equality.



Proofs by contradiction.

(*) Assume the opposite of what you want to show.

____.

End up showing the opposite of some assumption (not necessarily the (*) assumption). Contradiction.

Thus Assumption (*) is wrong and what you want to show is true.

Counterexamples.

To show that a statement, "If ______then _____", is false you *must* give an example.

To show: There exists a _____ such that

(a) it satisfies the ifs of the statement that you are showing is false.

(b) it satisfies the opposite of some assertion in the thens of the statement that you are showing is false.

Proofs of uniqueness.

To show that an object is unique you must show that if there are two of them then they are really the same.

```
To show: A THING is unique.
Assume X_1 and X_2 are both THINGs.
To show: X_1 = X_2.
```

Proofs by induction.

A statement to be proved by induction must have the form

If n is a positive integer then <u>A</u>.

The proof by induction should have the form

```
Proof by induction.

Base case:

To show: If n = 1 then <u>A</u>.

.

.

.

.

Thus if n = 1 then <u>A</u>.

Induction step:

Let N be a posititive integer and assume that if n is a positive integer and n < N then <u>A</u>.

To show: <u>A</u>.
```

This last to show line contains exactly the same statement A as in the original statement except with *n* replaced by *N*.

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