The Language of Mathematics

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1. The grammar of mathematics

- **Definitions** are the foundation of mathematics.
- **Theorems** are the landmarks of mathematics.
- **Proofs** are the explanation of mathematics.

Learning to read, write and speak mathematics is a skill that anyone can learn. Like all languages, it requires lots of practice to use it fluently.

Like all languages, the grammar of quality mathematical communication is very rigid. The grammar of a definition is:

```
A noun is a _____ such that
     (a) If _____ then _____, and
     (b) If _____ then _____, and
     (c) If _____ then _____, and ...
```

An adjective is most conveniently defined by putting it in the form of a noun:

```
An adjective noun is a _____ such that
     (a) If _____ then _____, and
     (b) If _____ then _____, and
     (c) If _____ then _____, and ...
```

Sometimes definitions of adjectives take the form:

```
Let S be a noun.
The noun S is adjective if S satisfies
     (a) If _____ then _____, and
     (b) If _____ then _____, and
     (c) If _____ then _____, and ...
```

The words "let" and "assume" are synonyms for "if". The grammar of a lemma, proposition or theorem (or any other statement) is:

```
If _____ then _____.
```

Two special constructions in mathematical language are:

```
There exists _____ such that _____.
```

and

```
There exists a unique _____ such that _____.
```

It is **impossible** to prove a statement without being able to write down the definitions of all the terms in the statement.

2. How to do proofs

There *is* a certain "formula" or method to doing proofs. Some of the guidelines are given below. The most important factor in learning to do proofs is practice, just as when one is learning a new language.

1. There are very few words needed in the structure of a proof. Organized in rows by synonyms they are:

```
To show
Assume, Let, Suppose, Define, If
Since, Because, By
Then, Thus, So
There exists, There is
Recall, We know, But
```

2. The overall structure of a proof is a block structure like an outline. For example:

```
To show: If __A__ then __B__ and __C__.
     Assume: __A__.
To show:
     (a) __B__.
     (b) __C__.
(a) To show: __B__.
          ______.
          ______.
          ______.
     Thus __B__.
(b) To show: __C__.
          ______.
          ______.
          ______.
     Thus __C__.
     So __B__ and __C__.
     So, if __A__ then __B__ and __C__.
```

3. Any proof or section of proof begins with one of the following:

(a) `To show: If __A__ then __B__.`
(b) `To show: There exists __C__ such that __D__.`
(c) `To show: __E__.`

4. Immediately following this, the next step is

Case (a) Assume the ifs and `to show' the thens. The next lines usually are

- `Assume __A__.`
- `To show: __B__.`

Case (b) To show an object exists you must find it. The next lines usually are

- `Define __xxx__.`
- `To show: __xxx__ satisfies __D__.`

Case (c) Rewrite the statement in __E__ by using a definition. The next line is usually

- To show __E'__.

A useful guideline is, "Don't think too much." Following the "method" usually produces a proof without thinking. Most of doing proofs is simply rewriting what has come just before in a different form by plugging in a definition.

There are some kinds of proofs which have a special structure.

Proofs of equality.

```
To show: __A=B__ .
     Left Hand side: __A = ···____
                     ______= ···__
                     ______= ···__
                     ______= ···__
                     ______= expression____.

     Right Hand side: __B = ···____
                     ______= ···__
                     ______= ···__
                     ______= ···__
                     ______= the same expression____.
```

Proofs by contradiction.

(*) `Assume` the opposite of what you want to show.

______.
______.
______.

End up showing the opposite of some assumption (not necessarily the (*) assumption).
`Contradiction.`
`Thus` Assumption (*) is wrong and what you want to show is true.

Counterexamples.

To show that a statement, "If ___ then ___", is false you *must* give an example.

`To show: There exists a ______ such that`
(a) it satisfies the `ifs` of the statement that you are showing is false.

(b) it satisfies the opposite of some assertion in the `thens` of the statement that you are showing is false.

Proofs of uniqueness.

To show that an object is unique you must show that if there are two of them then they are really the same.

To show: A *THING* is unique.
Assume X_1 and X_2 are both *THING*s.
To show: $X_1 = X_2$.

Proofs by induction.

A statement to be proved by induction must have the form

If n is a positive integer then __A__.

The proof by induction should have the form

Proof by induction.
Base case:
To show: If n = 1 then __A__.
______.
______.
______.
Thus if n = 1 then __A__.
Induction step:
Let N be a posititive integer and assume that if n is a positive integer and n < N then __A__.
To show: __A__.

This last to show line contains exactly the same statement `A` as in the original statement except with n replaced by N.