Representation Theory
Lecture Notes: Chapter 4

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Version: October 13, 2008

Abstract.

Spaces

A *topological space* is a set X with a specified collection of *open* subsets of X which is closed under unions, finite intersections, complements and contains $\emptyset$ and X. A *continuous function* $f: X \to Y$ is a map such that $f^{-1}(V)$ is open in X for all open subsets $V \subseteq Y$. The morphisms in the category of topological spaces are *continuous* functions.

(a) A *closed* subset of X is the complement of an open set of X.
(b) The space X is *compact* if every open cover has a finite subcover.
(c) The space X is *locally compact* if every point has a neighborhood with compact closure.
(d) The space X is *totally disconnected* if there is no connected subset with more than one element.
(e) The space X is *Hausdorff* if $\Delta_X = \{(x,x) \mid x \in X\}$ is a closed subspace of $X \times X$, where $X \times X$ has the product topolgy.

The topological space X is Hausdorff if and only if for any two points in X there exist neighborhoods of each of them that do not intersect.

A *metric space* is a set X with a metric $d: X \times X \to \mathbb{R}_{\geq 0}$ such that A *Cauchy sequence* is a sequence $(p_i \in V \mid i \in \mathbb{Z}_{>0})$ such that, for every positive real number ϵ there is a positive integer N such that $d(p_n, p_m) < \epsilon$ for all $m, n > N$. A sequence $(p_i \in V \mid i \in \mathbb{Z}_{>0})$ *converges* if there is a $p \in V$ such that, for every $\epsilon \in \mathbb{R}_{>0}$, there is an $N \in \mathbb{Z}_{>0}$ such that $d(p_n, p) < \epsilon$ for all $n > N$. A metric space is *complete* if all Cauchy sequences converge.

Sheaves

Let X be a topological space. A *sheaf* on X is a contravariant functor

$$\begin{array}{rccc} \mathcal{O}_X\colon & \{\text{open sets of } X\} & \longrightarrow & \{\text{rings}\} \\ & U & \longmapsto & \mathcal{O}_X(U) \end{array}$$

such that if $\{U_\alpha\}$ is an open cover of U and $f_\alpha \in \mathcal{O}_X(U_\alpha)$ are such that

$$f_\alpha\big|_{U_\alpha \cap U_\beta} = f_\beta\big|_{U_\alpha \cap U_\beta}, \qquad \text{for all } \alpha,\ \beta,$$

* Research supported in part by National Science Foundation grant DMS-9622985.

then there is a unique $f \in \mathcal{O}_X(U)$ such that $f_\alpha = f|_{U_\alpha}$ for all α. A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf on X. The *stalk* of $\mathcal{O}_X$ at $x \in X$ is

$$\mathcal{O}_{X,x} = \operatorname{ind}\lim_{U} \mathcal{O}_X(U),$$

where the limit is over all neighborhoods U of x.

Note: an alternate way of stating the condition in the definition of a sheaf is to say that the sequence

$$\mathcal{O} \to \mathcal{O}_x(U) \xrightarrow{i} \prod_\alpha \mathcal{O}_x(U_\alpha) \underset{k}{\overset{j}{\rightrightarrows}} \prod_{\alpha,\beta} \mathcal{O}_x(U_\alpha \cap U_\beta)$$

is exact where

i is the map induced by the inclusions $U_\alpha \hookrightarrow U$,
j is the map induced by the inclusions $U_\alpha \cap U_\beta \hookrightarrow U_\alpha$,
k is the map induced by the inclusions $U_\alpha \cap U_\beta \hookrightarrow U_\beta$,

and exactness of the sequence means $\mathrm{im} i = \ker(j - k)$.

Smooth manifolds

A *manifold* is a topological space X which is locally homeomorphic to $\mathbb{R}^n$. Locally homeomorphic to $\mathbb{R}^n$ means that for each $x \in X$ there is an open neighborhood U of x, an open set V in $\mathbb{R}^n$ and a homeomorphism $\phi: U \to V$. The map $\phi: U \to V$ is a *chart*. An *atlas* is an open covering (U_α) of X, a set of open sets (V_α) of $\mathbb{R}^n$ and a collection of charts $\phi_\alpha: U_\alpha \to V_\alpha$. Examples of manifolds are

PICTURE OF SPHERE — sphere

PICTURE OF TORUS — torus

A *smooth manifold* is a manifold with an atlas (ϕ_α) such that for each pair of charts ϕ_α, ϕ_β the maps

$$\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

are smooth (i.e. C^∞). Let M be a smooth manifold and let U be an open subset of M. The ring of smooth functions on U is the set of functions $f: U \to \mathbb{R}$ that are smooth at every point of U, i.e.

If $x \in U$ then there is a chart $\phi_\alpha: U_\alpha \to V_\alpha$ such that $x \in U_\alpha$ and

$$f \circ \phi_\alpha^{-1}: V_\alpha \to \mathbb{R}, \qquad \text{is } C^\infty.$$

Let V_α be an open set of $\mathbb{R}^n$. For each open set V of V_α let $C^\infty(V)$ be the set of functions $f: V \to \mathbb{R}$ that are C^∞ at every point of V. If $V \hookrightarrow V'$ then we have a map

$$\begin{array}{ccc} C^\infty(V') & \longrightarrow & C^\infty(V) \\ f & \longmapsto & f\big|_V . \end{array}$$

Thus

$$\begin{array}{cccc} C^\infty: & \{\text{open sets of } V_\alpha\} & \longrightarrow & \{\text{rings}\} \\ & V & \longmapsto & C^\infty(V) \end{array}$$

is a sheaf on V_α and (V_α, C^∞) is a ringed space.

A *smooth manifold* is a Hausdorff topological space which is locally isomorphic to $\mathbb{R}^n$, i.e. a Hausdorff ringed space (M, C^∞) with an open cover (U_α) such that each (U_α, C^∞) is isomorphic (as a ringed space) to an open set (V_α, C^∞) of $\mathbb{R}^n$.

Varieties

A *affine algebraic variety* over $\bar{\mathbb{F}}$ is a set

$$X = \{(x_1, \ldots, x_n) \mid f_\alpha(x_1, \ldots, x_n) = 0 \text{ for all } f_\alpha \in S\}$$

where S is a set of polynomials in $\bar{\mathbb{F}}[t_1, t_2, \ldots, t_n]$. By definition, these are the closed sets in the *Zariski topology* on $\bar{\mathbb{F}}^n$. Let U be an open set of X and define $\mathcal{O}_X(U)$ to be the set of functions $f: U \to \bar{\mathbb{F}}$ that are *regular* at every point of $x \in U$, i.e.

For each $x \in U$ there is a neigborhood $U_\alpha \subseteq U$ of x and functions $g, h \in \bar{F}[t_1, \ldots, t_n]$ such that $h(y) \neq 0$ and $f(y) = g(y)/h(y)$ for all $y \in U_\alpha$.

Then $\mathcal{O}_X$ is a sheaf on X and $(X, \mathcal{O}_X)$ is a ringed space. The sheaf $\mathcal{O}_X$ is the *structure sheaf* of the affine algebraic variety X.

A *variety* is a ringed space $(X, \mathcal{O})$ such that

(a) X has a finite open covering $\{U_\alpha\}$ such that each $(U_\alpha, \mathcal{O}|_{U_\alpha})$ is isomorphic to an affine algebriac variety,

(b) $(X, \mathcal{O})$ satisfies the *separation axiom*, i.e.

$$\Delta_X = \{(x, x) \mid x \in X\} \text{ is closed in } X \times X,$$

where the topology on $X \times X$ is the Zariski topology. (Note that the Zariski topology on $X \times X$ is, in general, finer than the product topology on $X \times X$.)

A *prevariety* is a ringed space which satisfies (a).

Schemes

Let A be a finitely generated commutative $\bar{\mathbb{F}}$-algebra and let

$$X = \mathrm{Hom}_{\bar{\mathbb{F}}\mathrm{alg}}(A, \bar{\mathbb{F}}).$$

By definition, the closed sets of X in the Zariski topology are the sets

$$C_J = \{M \in X \mid J \subseteq M\} \quad \text{for } J \subseteq A,$$

where we identify the points of X with the maximal ideals in A. Let U be an open set of X and let

$$\mathcal{O}_X(U) = \{g/h \mid g, h \in A,\ x(h) \neq 0 \text{ for all } x \in U\}.$$

Then $\mathcal{O}_X$ is a sheaf on X and $(X, \mathcal{O}_X)$ is a ringed space. The space X is an *affine* $\bar{\mathbb{F}}$*-scheme*.

An $\bar{\mathbb{F}}$*-variety* is a ringed space $(X, \mathcal{O}_X)$ such that

(a) For each $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring,

(b) X has a finite open covering $\{U_\alpha\}$ such that each $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is isomorphic to an affine $\bar{\mathbb{F}}$-scheme,

(c) $(X, \mathcal{O}_X)$ is *reduced*, i.e. for each $x \in X$ the local ring $\mathcal{O}_{X,x}$ has no nonzero nilpotent elements,

(d) $(X, \mathcal{O}_X)$ satisfies the *separation axiom*, i.e.

$$\Delta_X = \{(x, x) \mid x \in X\} \text{ is closed in } X \times X.$$

A *prevariety* is a ringed space which satisfies (a),(b) and (c). An $\bar{\mathbb{F}}$*-scheme* is a ringed space which satisfies (a) and (b). An $\bar{\mathbb{F}}$*-space* is a ringed space which satisfies (a).

Groups

A *group* is a set G with a mulitplication such that
(a) $(ab)c = a(bc)$, for all $a, b, c \in G$,
(b) There is an identity $1 \in G$,
(c) Every element of G is invertible. Let

$$[x, y] = xyx^{-1}y^{-1}, \qquad \text{for } x, y \in G.$$

The *lower central series* of G is the sequence

$$C^1(G) \supseteq C^2(G) \supseteq \cdots, \qquad \text{where } C^1(G) = G \text{ and } C^{i+1}(G) = [G, C^i(G)].$$

The *derived series* of G is the sequence

$$D^0(G) \supseteq D^2(G) \supseteq \cdots, \qquad \text{where } D^0(G) = G \text{ and } D^{i+1}(G) = [D^i(G), D^i(g)].$$

Let G be a group.
(a) G is *abelian* if $[G, G] = \{1\}$.
(b) G is *nilpotent* if $C^n(G) = \{1\}$ for all sufficiently large n.
(c) G is *solvable* if $D^n(G) = \{1\}$ for all sufficiently large n.

The *radical* $R(G)$ of a Lie group G is the largest connected solvable normal subgroup of G.

A *topological group* is a topological space G which is also a group such that multiplication and inversion

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (g, h) & \longmapsto & gh \end{array} \qquad \begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & g^{-1} \end{array}$$

are morphisms of topological spaces, i.e. continuous maps.
A *Lie group* is a smooth manifold with a group structure such that multiplication and inversion are morphisms of smooth manifolds, i.e. smooth maps.
A *complex Lie group* is a complex analytic manifold which is also a group such that multiplication and inversion are morphisms of complex analytic manifolds, i.e. holomorphic maps.
A *linear algebraic group* is an affine algebraic variety which is also a group such that multiplication and inversion are morphisms of affine algebraic varieties.
A *group scheme* is a scheme which is also a group such that multiplication and inversion are morphisms of schemes.

Lie groups

The Lie group $S^1 = \mathbb{R}/\mathbb{Z} = U_1(\mathbb{C})$. A *torus* is a Lie group G is isomorphic to $S^1 \times \cdots S^1$ (k factors), for some $k \in \mathbb{Z}_{>0}$.

A connected Lie group is *semisimple* if $R(G) = \{1\}$.

Let G be a Lie group and let $x \in G$. A *tangent vector* at x is a linear map $\xi_x : C^\infty(G) \to \mathbb{R}$ such that

$$\xi_x(f_1 f_2) = \xi_x(f_1) f_2(x) + f_1(x)\xi_x(f_2), \qquad \text{for all } f_1, f_2 \in C^\infty(G).$$

A *vector field* is a linear map $\xi : C^\infty(G) \to C^\infty(G)$ such that

$$\xi(f_1 f_2) = \xi(f_1) f_2 + f_1 \xi(f_2), \qquad \text{for } f_1, f_2 \in C^\infty(G).$$

A *left invariant* vector field on G is a vector field $\xi : C^\infty(G) \to C^\infty(G)$ such that

$$L_g \xi = \xi L_g, \qquad \text{for all } g \in G.$$

A *one parameter subgroup* of G is a smooth group homomorphism $\gamma: \mathbb{R} \to G$. If γ is a one parameter subgroup of G define

$$\frac{d}{dt} f(\gamma(t)) = \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}.$$

The following proposition says that we can identify the three vector spaces

(1) {left invariant vector fields on G},
(2) {one parameter subgroups of G},
(3) {tangent vectors at $1 \in G$}.

Proposition 0.1. *The maps*

$$\begin{array}{ccc} \{\textit{left invariant vector fields}\} & \longrightarrow & \{\textit{tangent vectors at } 1\} \\ \xi & \longmapsto & \xi_1 \end{array}$$

and

$$\begin{array}{ccc} \{\textit{one paramemeter subgroups}\} & \longrightarrow & \{\textit{tangent vectors at } 1\} \\ \gamma & \longmapsto & \gamma_1 \end{array}$$

where

$$\xi_1 f = (\xi f)(1), \qquad \textit{and} \qquad \gamma_1 = \left(\frac{d}{dt} f(\gamma(t))\right)\Big|_{t=0},$$

are vector space isomorphisms.

The *Lie algebra* $\mathfrak{g} = Lie(G)$ of the Lie group G is the tangent space to G at the identity with the bracket $[,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by

$$[\xi_1, xi_2] = \xi_1\xi_2 - \xi_2\xi_1, \qquad \text{for} \quad \xi_1, \xi_2 \in \mathfrak{g}.$$

Let $\phi: G \to H$ be a Lie group homomorphism and let $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$. Then

$$\begin{array}{ccc} C^\infty(H) & \xrightarrow{\phi^*} & C^\infty(G) \\ f & \longmapsto & f \circ \phi \end{array}$$

and the *differential* of ϕ is the Lie group homomorphism $\mathfrak{g} \xrightarrow{d\phi} \mathfrak{h}$ given by

$$\begin{aligned} d\phi(\xi_1) &= \xi_1 \circ \phi^*, && \text{if } \xi_1 \text{ is a tangent vectors at the identity,} \\ d\phi(\xi) &= \xi \circ \phi^*, && \text{if } \xi \text{ is a left invariant vector field,} \\ d\phi(\gamma) &= \phi \circ \gamma, && \text{if } \gamma \text{ is a one parameter subgroup.} \end{aligned}$$

(Note: It should be checked that (a) the map $d\phi$ is well defined, (b) the three definitions of $d\phi$ are the same, and (c) that $d\phi$ is a Lie algebra homorphisms. These checks are not immediate, but are straightforward manipulations of the definitions.) The map

$$\begin{array}{ccc} \text{the category of Lie groups} & \longrightarrow & \text{the category of Lie algebras} \\ G & \longmapsto & \text{Lie(G)} \\ \phi & \longmapsto & d\phi \end{array}$$

is a functor. This functor is *not* one-to-one; for example, the Lie groups $O_n(\mathbb{R})$ and $SO_n(\mathbb{R})$ have the same Lie algebra. On the other hand, the Lie algebra contains the structure of the Lie groups in a neighborhood of the identity. The *exponential map* is

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & G \\ tX & \longmapsto & e^{tX}, \end{array} \qquad \text{where} \quad e^t X = \gamma(t)$$

is the one parameter subgroup corresponding to $X \in \mathfrak{g}$. This map is a homeomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of 1 in G.

Theorem 0.2. *(Lie's theorem) The functor*

$$\begin{array}{rccc} \mathrm{Lie}\colon & \{\textit{connected simply connected Lie groups}\} & \longrightarrow & \{\textit{Lie algebras}\} \\ & G & \longmapsto & \mathfrak{g} = Lie(G) = T_1(G) \end{array}$$

is an equivalence of categories.

If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n$ then the matrices

$$\{e^{tX} \mid t \in \mathbb{R}, X \in \mathfrak{gl}_n\}, \quad \text{where} \quad e^t X = \sum_{k\geq 0} \frac{t^k X^k}{k!},$$

form a group with Lie algebra $\mathfrak{g}$.

$$\begin{aligned} e^{tX}e^{tY} &= e^{t(X+Y)+(t^2/2)[X,Y]+\cdots}, \\ e^{tX}e^{tY}e^{-tX} &= e^{tY+t^2[X,Y]+\cdots}, \\ e^{tX}e^{tY}e^{-tX}e^{-tY} &= e^{t^2[X,Y]+\cdots}, \end{aligned}$$

Let G be a Lie group and let $\mathfrak{g} = \mathrm{Lie}(G)$. Let $x \in G$. Then the differential of the Lie group homomorphism

$$\begin{array}{rccc} \mathrm{Int}_x\colon & G & \longrightarrow & G \\ & g & \longmapsto & xgx^{-1} \end{array}$$

is a Lie algebra homomorphism

$$\mathrm{Ad}_x\colon \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Since there is a map Ad_x for each $x \in G$, there is a map

$$\begin{array}{rccc} \mathrm{Ad}\colon & G & \longrightarrow & GL(\mathfrak{g}) \\ & x & \longmapsto & \mathrm{Ad}_x \end{array} \qquad \text{and} \qquad \mathrm{Ad}_x\mathrm{Ad}_y = \mathrm{Ad}_{xy}, \quad \text{for } x, y \in G,$$

since $\mathrm{Int}_x\mathrm{Int}_y = \mathrm{Int}_{\mathrm{xy}}$. The differential of Ad is

$$\begin{array}{rccc} \mathrm{ad}\colon & \mathfrak{g} & \longrightarrow & \mathrm{End}(\mathfrak{g}) \\ & X & \longmapsto & \mathrm{ad}_X \end{array}, \qquad \text{where} \qquad \begin{array}{rccc} \mathrm{ad}_X\colon & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & Y & \longmapsto & [X,Y] \end{array},$$

since

$$\frac{d}{dt}\frac{d}{ds} e^{tX}e^{sY}e^{-tX}\Big|_{s=0,t=0} = [X,Y], \qquad \text{for } X, Y \in \mathfrak{g}.$$

Define a (right) action of G on $C^\infty(G)$ by

$$(R_x f)(g) = f(gx), \qquad \text{for } x \in G, f \in C^\infty(G), g \in G.$$

Then

$$\mathrm{Ad}_x\xi = R_x\xi R_{x^{-1}}, \qquad \text{for all } x \in G,\ \xi \in \mathfrak{g},$$

since, for $x \in G$, $\mathrm{Int}_x^*(\mathrm{Ad}_x\xi) = \xi \circ \mathrm{Int}_x^* = \xi L_{x^{-1}}R_{x^{-1}} = L_{x^{-1}}\xi R_{x^{-1}}L_{x^{-1}}R_{x^{-1}}R_x\xi R_{x^{-1}} = \mathrm{Int}_x^*(R_x\xi R_{x^{-1}})$.

Recall that the adjoint representation of G is

$$\begin{array}{rccc} \mathrm{Ad}\colon & G & \longrightarrow & GL(\mathfrak{g}) \\ & x & \longmapsto & \mathrm{Ad}_x \end{array} \quad \text{where} \quad \begin{array}{rccc} \mathrm{Ad}_x\colon & \mathfrak{g} & \longrightarrow & fg \\ & \xi & \longmapsto & R_x \xi R_{x^{-1}} \end{array}$$

is the differential of

$$\begin{array}{rccc} \mathrm{Int}_x\colon & G & \longrightarrow & G \\ & g & \longmapsto & xgx^{-1} \end{array}.$$

The *coadjoint representation* of G is the dual of the adjoint representation, i.e. the action of G on $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$ given by

$$(g\phi)(X) = \phi(\mathrm{Ad}_{g^{-1}} X), \qquad \text{for } g \in G,\ \phi \in \mathfrak{g}^*,\ X \in \mathfrak{g}.$$

A *coadjoint orbit* is the set produced by the action of G on an element $\phi \in \mathfrak{g}^*$, i.e. $G\phi \subseteq \mathfrak{g}^*$ is a coadjoint orbit.

Let G be a Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Then G^0 is nilpotent if and only if $Lie(G)$ is nilpotent, and G^0 is solvable if and only if $Lie(G)$ is solvable. A *semisimple Lie group* is a connected Lie group with semisimple Lie algebra.

The class of *reductive* Lie groups is the largest class of Lie groups which contains all the semisimple Lie groups and parabolic subgroups of them and for which the representation theory is still controllable. A real Lie group is *reductive* if there is a linear algebraic group G over $\mathbb{R}$ whose identity component (in the Zariski topology) is reductive and a morphism $\nu\colon G \to G(\mathbb{R})$ with finite kernel, whose image is an open subgroup of $G(\mathbb{R})$. For the definition of *Harish-Chandra* class see Knapp's article.

(a) $U(n) = \{x \in M_n(\mathbb{C}) \mid x\bar{x}^t = \mathrm{id}\}$.
(b) $Sp(2n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A^t J A = J\}$.
(c) $Sp_{2n} = Sp(2n, \mathbb{C}) \cap U(2n)$.

Theorem 0.3. *The simple compact Lie groups are*

(a) (Type A) $SU_n(\mathbb{C})$
(b) (Type B_n) $S0_{2n+1}(\mathbb{R})$, $n \geq$
(c) (Type C_n) $Sp_{2n}(\mathbb{C}) \cap U_n$, $n \geq 1$,
(d) (Type D_n) $SO_{2n}(\mathbb{R})$, $n \geq 4$,
(e) ???

Theorem 0.4. *If G is a Lie group such that G/G^0 is finite then*

(a) G has a maximal compact subgroup,
(b) Any two maximal compact subgroups are conjugate,
(c) G is homeomorphic to $K \times \mathbb{R}^m$ under the map

$$\begin{array}{ccc} K \times \mathfrak{p} & \longrightarrow & G \\ (k, x) & \longmapsto & ke^x \end{array}$$

where K is a maximal compact subgroup of G and $\mathfrak{p}$ =??????????.

(d) If G is a semisimple Lie group then

$$K = \{g \in G \mid \Theta(g) = g\},$$

where Θ is the Cartan involution on G, is a maximal compact subgroup of G. For matrix groups

$$\begin{array}{cccc} \Theta\colon & G & \longrightarrow & G \\ & g & \longmapsto & \overline{(g^{-1})^t} \end{array}$$

is the Cartan involution.

On the Lie algebra level

$$\begin{array}{cccc} \theta\colon & \mathfrak{g} & \longrightarrow & fg \\ & x & \longmapsto & -\bar{x}^t \end{array} \qquad \mathfrak{k} = \{x \in \mathfrak{g} \mid \theta x = x\}, \qquad \mathfrak{p} = \{s \in \mathfrak{g} \mid \theta x = -x\},$$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}, \qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}.$$

Theorem 0.5. *There is an equivalence of categories*

$$\begin{array}{ccc} \{\textit{compact connected Lie groups}\} & \longleftrightarrow & \{\textit{connected reductive algebraic groups over } \mathbb{C}\} \\ U & \longleftrightarrow G & \end{array}$$

where U is the maximal compact subgroup of G and G is the algebraic group with coordinate ring $C(U)^{\mathrm{rep}}$. The group G is the complexification of U.
(b) The functor

$$\mathrm{Res}^G_K\colon \{\textit{holomorphic representations of } G\} \longrightarrow \{\textit{representations of } K\}$$

is an equivalence of categories.

Proof. (a) The point of (a) is that for compact grops the continuous functions separate the points of G and for algebraic groups the polynomial functions separate the points of G, and, for $\mathbb{C}$ and $\mathbb{R}$ the polynomial functions are dense in the continuous functions.

Examples: Under the equivalence of (???)
(a) semisimple algebraic groups correspond exactly to the Lie groups with finite center,
(b) algebraic tori correspond exactly to geometric tori.
(c) irreducible finite dimensional representations of G correspond exactly to irreducible finite dimensional representations of U.

$$\begin{array}{ccc} U_n & \longleftrightarrow & GL_n(\mathbb{C}) \\ SU_n & \longleftrightarrow & SL_n(\mathbb{C}) \\ SO_{2n+1}(\mathbb{R}) & \longleftrightarrow & SO_{2n+1}(\mathbb{C}) \\ Sp_{2n} & \longleftrightarrow & Sp_{2n}(\mathbb{C}) \\ SO_{2n}(\mathbb{R}) & \longleftrightarrow & SO_{2n}(\mathbb{C}) \end{array}$$

Other examples are $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $PGL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $SO_n(\mathbb{C})$, Pin_n, Spin_n, $Sp_{2n}(\mathbb{C})$, $PSp_{2n}(\mathbb{C})$, $U_n(\mathbb{C})$, $SU_n(\mathbb{C})$, $U_n(\mathbb{C})/Z(U_n(\mathbb{C}))$, $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$, ….

Equivalences:

$$\begin{aligned} \{\text{compact Lie groups}\} &\longleftrightarrow \{\text{complex semisimple Lie groups}\} \\ &\longleftrightarrow \{\text{semisimple algebraic groups}\} \\ &\longrightarrow \{\text{complex semisimple Lie algebras}\} \end{aligned}$$

A *representation* of G is an action of G on a vector space by linear transformations. The words representation and G-*module* are used interchangably. A *complex representation* is a representation where V is a vector space over $\mathbb{C}$. In order to distinguish the group element g from the linear transformation of V given by the action of g write $V(g)$ for the linear transformation. Then

$$V: G \longrightarrow GL(V)$$

and the statement that the representation is a group action means

$$V(xy) = V(x)V(y), \qquad \text{for all } x, y \in G.$$

Unless otherwise stated we shall assume that all representations of G are Lie group homomorphisms. A *holomorphic representation* is a representation in the category of complex Lie groups.

A representation is *irreducible*, or *simple*, if it has no subrepresentations (except 0 and itself). In the case when V is a topological vector space then a subrepresentation is required to be a closed subspace of V. The *trivial* G-module is the representation

$$\begin{array}{rccl} \mathbf{1}\colon & G & \longrightarrow & \mathbb{C}^* = GL_1(\mathbb{C}) \\ & g & \longmapsto & 1 \end{array}$$

If V and W are G-modules the *tensor product* is the action of G on $V \otimes W$ given by

$$g(v \otimes w) = gv \otimes gw, \qquad \text{for } v \in V,\ w \in W,\ g \in G.$$

If V is a G-module the *dual* G-module to V is the action of G on $V^* = \mathrm{Hom}(V, \mathbb{C})$ (linear maps $\psi: V \to \mathbb{C}$) given by

$$(g\psi)(v) = \psi(g^{-1}v), \qquad \text{for} \quad g \in G, \psi \in V^*, v \in V.$$

The maps

$$\begin{array}{ccc} \mathbf{1} \otimes V & \xrightarrow{\sim} & V \\ 1 \otimes v & \longmapsto & v \end{array} \quad \text{and} \quad \begin{array}{ccc} V \otimes \mathbf{1} & \xrightarrow{\sim} & V \\ v \otimes 1 & \longmapsto & v \end{array}$$

are G-module isomorphisms for any V. The maps

$$\begin{array}{ccc} V^* \otimes V & \longrightarrow & \mathbf{1} \\ \phi \otimes v & \longmapsto & \phi(v) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{1} & \longrightarrow & V \otimes V^* \\ 1 & \longmapsto & \sum_i b_i \otimes \beta_i^* \end{array}$$

where $\{b_i\}$ is a basis of V and $\{\beta_i^*\}$ is the dual basis in V^* are G-module homomorphisms.

If $V: G \to GL(V)$ is a homomorphism of Lie groups then the differential of V is a map

$$dV: \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

which satisfies

$$dV([x, y]) = [dV(x), dV(y)] = dV(x)dV(y) - dV(y)dV(x),$$

for $x, y \in \mathfrak{g}$. A *representation* of a Lie algebra $\mathfrak{g}$, or $\mathfrak{g}$-*module*, is an action of $\mathfrak{g}$ on a vector space V by linear transformations, i.e. a linear map $\phi: \mathfrak{g} \to \mathrm{End}(V)$ such that

$$V([x, y]) = [V(x), V(y)] = V(x)V(y) - V(y)V(x), \qquad \text{for all } x, y \in \mathfrak{g},$$

where $V(x)$ is the linear transformation of V determined by the action of $x \in \mathfrak{g}$. The *trivial representation* of $\mathfrak{g}$ is the map

$$\begin{array}{rcl} \mathbf{1}\colon \quad \mathfrak{g} & \longrightarrow & \mathbb{C} \\ x & \longmapsto & 0 \end{array}$$

If V is a $\mathfrak{g}$-module, the *dual* $\mathfrak{g}$-module is the $\mathfrak{g}$-action on $V^* = \mathrm{Hom}(V, \mathbb{C})$ given by

$$(x\phi)(v) = \phi(-xv), \quad \text{for } x \in \mathfrak{g},\ \phi \in V^*,\ v \in V.$$

If V and W are $\mathfrak{g}$-modules the *tensor product* of V and W is the $\mathfrak{g}$-action on $V \otimes W$ given by

$$x(v \otimes w) = xv \otimes w + v \otimes xw, \qquad x \in \mathfrak{g}, v \in V, w \in W.$$

The definitions of the trivial, dual and tensor product $\mathfrak{g}$-modules are accounted for by the following formulas:

$$\begin{aligned} \frac{d}{dt} 1\big|_{t=0} &= \frac{d}{dt} e^{t\cdot 0}\big|_{t=0} = 0, \\ \frac{d}{dt}(e^{tX})^{-1}\big|_{t=0} &= \frac{d}{dt} e^{-tX}\big|_{t=0} = -X, \\ \frac{d}{dt}(e^{tX} \otimes e^{tX})\big|_{t=0} &= \frac{d}{dt}(1 + tX + \frac{t^2X^2}{2!} + \cdots) \otimes (1 + tX + \frac{t^2X^2}{2!} + \cdots)\big|_{t=0} \\ &= \frac{d}{dt}(1 \otimes 1 + t(X \otimes 1 + 1 \otimes X) + \cdots)\big|_{t=0} \\ &= X \otimes 1 + 1 \otimes X. \end{aligned}$$

Lie algebras

A *Lie algebra* over a field F is a vector space $\mathfrak{g}$ over F with a *bracket* $[,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is bilinear and satisfies

(1) $[x, y] = -[y, x]$, for all $x, y \in \mathfrak{g}$,
(2) (The Jacobi identity) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, for all $x, y, z \in \mathfrak{g}$.

The *derived series* of $\mathfrak{g}$ is the sequence

$$D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq \cdots, \qquad \text{where } D^0\mathfrak{g} = \mathfrak{g} \text{ and } D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}].$$

The *lower central series* of $\mathfrak{g}$ is the sequence

$$C^1\mathfrak{g} \supseteq C^2\mathfrak{g} \supseteq \cdots, \qquad \text{where } C^0\mathfrak{g} = \mathfrak{g} \text{ and } C^{i+1}\mathfrak{g} = [\mathfrak{g}, C^i\mathfrak{g}].$$

Let $\mathfrak{g}$ be a Lie algebra.

(a) $\mathfrak{g}$ is *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.
(b) $\mathfrak{g}$ is *nilpotent* if $C^n(\mathfrak{g}) = 0$ for all sufficiently large n.
(c) $\mathfrak{g}$ is *solvable* if $D^n(\mathfrak{g}) = 0$ for all sufficiently large n.
(d) The *radical* $\mathrm{rad}(\mathfrak{g})$ is the largest solvable ideal of $\mathfrak{g}$.
(e) The *nilradical* $\mathrm{nil}(\mathfrak{g})$ is the largest nilpotent ideal????????? of $\mathfrak{g}$.
(f) $\mathfrak{g}$ is *semisimple* if $\mathrm{rad}(\mathfrak{g}) = 0$.
(g) $\mathfrak{g}$ is *reductive* if $\mathrm{nil}(\mathfrak{g}) = 0$. $\mathfrak{g}$ is *reductive* if all its representations are completely decomposable. $\mathfrak{g}$ is *reductive* if $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.
(h) A *Cartan subalgebra* is a maximal abelian subalgebra of semisimple elements.

Then

$$0 \subseteq \mathrm{nil}(\mathfrak{g}) \subseteq \mathrm{rad}(\mathfrak{g}) \subseteq \mathfrak{g}$$

where $\mathrm{nil}(\mathfrak{g})$ is nilpotent, $\mathrm{rad}(\mathfrak{g})$ is solvable, $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple, $\mathrm{rad}(\mathfrak{g})/\mathrm{nil}(\mathfrak{g})$ is abelian, and $\mathrm{nil}(\mathfrak{g})$ is nilpotent.

Example. [Bou, Chap. I, §4, Prop. 5] The following are equivalent:

(a) $\mathfrak{g}$ is reductive,
(b) The adjoint representation of $\mathfrak{g}$ is semisimple,
(c) $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra,
(d) $\mathfrak{g}$ is the direct sum of a semisimple Lie algebra and a commutative Lie algebra.
(e) $\mathfrak{g}$ has a finite dimensional representation such that the associated bilinear form is nondegenerate.
(f) $\mathfrak{g}$ has a faithful finite dimensional representation.
(g) $\mathrm{rad}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

Theorem 0.6. *The finite dimensional simple Lie algebras over $\mathbb{C}$ are*

(a) (Type A_{n-1}) $\mathfrak{sl}_n(\mathbb{C})$, $n \geq 2$,
(b) (Type B_n) $\mathfrak{so}_{2n+1}(\mathbb{C})$, $n \geq 1$,
(c) (Type C_n) $\mathfrak{sp}_{2n}(\mathbb{C})$, $n \geq 1$,
(d) (Type D_n) $\mathfrak{so}_{2n}(\mathbb{C})$, $n \geq 4$, and
(e) the five simple Lie algebras E_6, E_7, E_8, F_4, G_2.

Theorem 0.7. *The finite dimensional simple Lie algebras over $\mathbb{R}$ are ?????*

Linear algebraic groups

A *linear algebraic group* is an afine algebraic variety G which is also a group such that multiplication and inversion are morphisms of algebraic varieties.

The following fundamental theorem is reason for the terminology *linear* algebraic group.

Theorem 0.8. *If G is a linear algebraic group then there is an injective morphism of algebraic groups $i: G \to GL_n(F)$ for some $n \in \mathbb{Z}_{>0}$.*

The *multiplicative group* is the linear algebraic group $\mathbb{G}_m = F^*$.

A matrix $x \in M_n(F)$ is

(a) *semisimple* if it is conjugate to a diagonal matrix,
(b) *nilpotent* if all it eigenvalues are 0, or, equivalently, if $x^n = 0$ for some $n \in \mathbb{Z}_{>0}$,
(c) *unipotent* if all its eigenvalues are 1, or equivalently, if $x - 1$ is nilpotent.

Let G be an linear algebraic group and let $i: G \to GL_n(F)$ be an injective homomorphism. An element $g \in G$ is

(a) *semisimple* if $i(g)$ is semisimple in $GL_n(F)$,
(b) *unipotent* if $i(g)$ is unipotent in $GL_n(F)$.

The resulting notions of semisimple and unipotent elements in G do not depend on the choice of the imbedding $i: G \to GL_n(\mathbb{C})$.

Theorem 0.9. *(Jordan decomposition) Let G be a linear algebraic group and let $g \in G$. Then there exist unique $g_s, g_u \in G$ such that*

(a) g_s *is semisimple,*
(b) g_u *is unipotent,*
(c) $g = g_s g_u = g_u g_s$.

Let G be a linear algebraic group.

(a) The *radical* $R(G)$ is the unique maximal closed connected solvable normal subgroup of G.
(b) The *unipotent radical* $R_u(G)$ is the unique maximal closed connected unipotent normal subgroup of G.
(c) G is *semisimple* if $R(G) = 1$.
(d) G is *reductive* if $R_u(G) = 1$. G is *reductive* if its Lie algebra is reductive.
(e) G is an *(algebraic) torus* if G is isomorphic to $\mathbb{G}_m \times \cdots \mathbb{G}_m$ (k factors) for some $k \in \mathbb{Z}_{>0}$.
(f) A *Borel subgroup* of G is a maximal connected closed solvable subgroup of G^0.

Let G be a linear algebraic group and let G^0 be the connected component of the identity in G. Then

$$1 \subseteq R_u(G) \subseteq R(G) \subseteq G^0 \subseteq G$$

where $R_u(G)$ is unipotent, $R(G)$ is solvable, G^0 is connected, G/G^0 is finite, $G^0/R(G)$ is semisimple, $R(G)/R_u(G)$ is a torus, and $R_u(G)$ is unipotent.

A linear algebraic group is *simple* if it has no proper closed connected normal subgroups. This implies that proper normal subgroups are finite subgroups of the center.

Proposition 0.10. *Let G be an algebraic group.*

(a) If G is nilpotent then $G \cong TU$ where T is a torus and U is unipotent.
(b) If G is connected reductive then $G = [G, G]Z^\circ$, where $[G, G]$ is semisimple and $[G, G] \cap Z^\circ$ is finite.
(c) If $[G, G]$ is semisimple then G is an almost direct product of simple groups, i.e. there are closed normal subgroups $G_1, \ldots, G_k$ in G such that $G = G_1 \cdot G_2 \cdots G_k$ and $G_i \cap (G_1 \cdots \hat{G}_i \cdots G_k$ is finite.

Example. If $G = GL_n(\mathbb{C})$ then

$$[G, G] = SL_n(\mathbb{C}), \qquad Z^\circ = \mathbb{C} \cdot \mathrm{Id}, \qquad \text{and} \qquad [G, G] \cap Z^\circ = \{\lambda \cdot \mathrm{Id} \mid \lambda^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}.$$

Structure of a simple algebraic group

$$x_\alpha(t) = e^{tX_\alpha}, \qquad w_\alpha(t) = x_\alpha(t) x_{-\alpha}(t^{-1}) x_\alpha(t), \qquad h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1},$$

$$U = \langle x_\alpha(t) \mid \alpha > 0 \rangle, \qquad T = \langle h_\alpha(t) \rangle \qquad N = \langle w_\alpha(t) \rangle \qquad B = TU \qquad W = N/T$$

The Langlands decomposition of a parabolic is $P = MAN$ where

$$M = \begin{pmatrix} A_1 & & & & \\ & A_2 & & 0 & \\ & & \ddots & & \\ & 0 & & A_{\ell-1} & \\ & & & & A_\ell \end{pmatrix}, \qquad \det(A_i) = 1,$$

$$A = \begin{pmatrix} a_1\mathrm{Id} & & & & \\ & a_2\mathrm{Id} & & 0 & \\ & & \ddots & & \\ & 0 & & a_{\ell-1}\mathrm{Id} & \\ & & & & a_\ell\mathrm{Id} \end{pmatrix}, \qquad a_i > 0,$$

$$N = \begin{pmatrix} \mathrm{Id} & & & & \\ & \mathrm{Id} & & * & \\ & & \ddots & & \\ & 0 & & \mathrm{Id} & \\ & & & & \mathrm{Id} \end{pmatrix},$$

and there is a corresponding decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ at the Lie algebra level.
The *Iwasawa decomposition* of G is $G = KAN$ where

$$K = \text{a maximal compact subgroup of } G,$$

$$A = \begin{pmatrix} a_1 & & & & \\ & a_2 & & 0 & \\ & & \ddots & & \\ & 0 & & a_{\ell-1} & \\ & & & & a_\ell\mathrm{Id} \end{pmatrix}, \qquad \det(A_i) = 1,$$

$$N = \begin{pmatrix} 1 & & & & \\ & 1 & & * & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix},$$

and the corresponding Lie algebra decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \text{where} \qquad \begin{array}{l} \mathfrak{k} = \{x \in \mathfrak{g} \mid \theta x = x\}, \qquad \mathfrak{p} = \{x \in \mathfrak{g} \mid \theta x = -x\}, \\ \mathfrak{a} = \text{a maximal abelian subspace of } \mathfrak{p}, \\ \mathfrak{n} = \text{the set of positive roots with respect to } \mathfrak{a}. \end{array}$$

The *Cartan decomposition* of G is $G = KAK$. The *Bruhat decomposition* of G is $G = BWB$.

Let $\mathfrak{g}$ be a semsimple complex Lie algebra.
(a) There is an involutory semiautomorphism σ_0 of $\mathfrak{g}$ (relative to complex conjugation) such that

$$\sigma_0(X_\alpha) = -X_\alpha, \qquad \sigma_0(H_\alpha) = -H_\alpha, \qquad \text{for all } \alpha \in R.$$

Let G be a Chevalley group over $\mathbb{C}$ viewed as a (real) Lie group.

(b) There is an (analytic) automorphism σ of G such that
$$\sigma x_\alpha(t) = x_{-\alpha}(-\bar{t}), \qquad \sigma(h_\alpha(t) = h_\alpha(\bar{t}^{-1}), \qquad \text{for all } \alpha \in R,\ t \in \mathbb{C}.$$
(c) A maximal compact subgroup of G is
$$K = \{g \in G \mid \sigma(g) = g\}.$$
(d) K is semisimple and connected.
(e) The Iwasawa decomposition is $G = BK$.
(f) The *Cartan decomposition* is $G = KAK$ where
$$A = \{h \in H \mid \mu(h) > 0 \text{ for all } \mu \in L\}.$$

Let Θ be a P.I.D., k the quotient field, and Θ^* the group of units of Θ (examples: $\Theta = \mathbb{Z}$, $\Theta = F[t]$, $\Theta = \mathbb{Z}_p$). If G is a Chevalley group over k let G_Θ be the subgroup of G with coordinates relative to M in Θ. Now let G be a semisimple Chevalley group over k.
(a) The *Iwasawa decomposition* is $G = BK$ where
$$K = G_\Theta.$$
(b) The *Cartan decomposition* is KA^+K where
$$A^+ = \{h \in H \mid \alpha(h) \in \Theta \text{ for all } \alpha \in R^+\}.$$
(c) If Θ is not a field (in particular if $\Theta = \mathbb{Z}$) then K is maximal in its commensurability class.
(d) If $\Theta = \mathbb{Z}_p$ and $k = \mathbb{Q}_p$ the K is a maximal compact subgroup in the p-adic topology.
(e) If Θ is a local PID and p is its unique prime then
  (1) The *Iwahori subgroup* $I = U_p^- H_\Theta U_\Theta$ is a subgroup of K.
  (2) $K = \bigcup_{w \in W} IwI$.
  (3) $IwI = IwU_{w,\Theta}$ with the last component determined uniquely mod $U_{w,p}$.

Classification Theorems

$$\{\text{semisimple algebraic groups over } \mathbb{C}\} \overset{1-1}{\longleftrightarrow} \{\text{lattices and root systems}\}$$

$$\{\text{complex semisimple Lie groups}\} \overset{1-1}{\longleftrightarrow} \{\text{semisimple algebraic groups over } \mathbb{C}\}$$

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{connected reductive}\\ \text{algebraic groups over } \mathbb{C}\end{matrix}\right\} & \overset{1-1}{\longleftrightarrow} & \{\text{compact connected Lie groups}\} \\
G & \longmapsto & U = \text{maximal compact subgroup of } G \\
\text{semsimple} & \longmapsto & \text{finite center} \\
\text{algebraic torus} & \longmapsto & \text{geometric torus}
\end{array}$$

$$\{\text{connected simply connected Lie groups}\} \overset{1-1}{\longleftrightarrow} \{\text{finite dimensional real Lie algebras}\}$$

$$\left\{\begin{matrix}\text{finite dimensional}\\ \text{complex simple Lie algebras}\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{Root systems:}\\ \text{4 infinite families and 5 exceptionals}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{finite dimensional}\\ \text{real simple Lie algebras}\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \{\text{12 infinite families and 23 exceptionals}\}$$

Functions, measures and distributions

Let G be a locally compact Hausdorff topological group and let μ be a Haar measure on G. The *support* of a function f is

$$\mathrm{supp}\, f = \{g \in G \mid f(g) \neq 0\}.$$

If it exists, the *convolution* of functions $f_1: G \to \mathbb{C}$ and $f_2: G \to \mathbb{C}$ is the function $(f_1 * f_2): G \to \mathbb{C}$ given by

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) d\mu(g). \tag{0.11}$$

Define an involution on functions $f: G \to \mathbb{C}$ by

$$f^*(g) = f(g^{-1}), \qquad \text{for all } g \in G.$$

Useful norms on functions $f: G \to \mathbb{C}$ are defined by

$$\begin{aligned} \|f\|_1 &= \int_G |f(g)| d\mu(g), \\ \|f\|_2^2 &= \int_G |f(g)|^2 d\mu(g), \\ \|f\|_\infty &= \sup\{|f(g)| \mid g \in G\}, \end{aligned}$$

If it exists, the *inner product* of functions $f_1: G \to \mathbb{C}$ and $f_2: G \to \mathbb{C}$ is

$$\langle f_1, f_2 \rangle \int_G f_1(g) \overline{f_2(g^{-1})} d\mu(g).$$

The left and right actions of G on functions $f: G \to \mathbb{C}$ are defined by

$$(L_g f)(x) = f(g^{-1}x), \qquad \text{and} \qquad (R_g f)(x) = f(xg), \qquad g, x \in G.$$

Some space of functions are

$\mathbb{C}G = \{\text{functions } f: G \to \mathbb{C} \text{ with finite support}\}$.

$\ell^1(G) = \{\text{functions } f: G \to \mathbb{C} \text{ with countable support and } \|f\| = \sum_{g\in G} |f(g)| < \infty\}$.

$L^1(G, \mu) = \{\text{functions } f: G \to \mathbb{C} \text{ such that } \|f\| = \int_G |f(g)| d\mu(g) < \infty.\}$.

Let X be a topological space. A *σ-algebra* is a collection of subsets of X which is closed under countable unions and complements and contains the set X. A *Borel set* is a set in the smallest σ-algebra $\mathcal{B}$ containing all open sets of X. A *Borel measure* is a function $\mu: \mathcal{B} \to [0, \infty]$ which is *countably additive*, i.e.

$$\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),$$

for every disjoint collection of A_i from $\mathcal{B}$. A *regular* Borel measure is a Borel measure which satisfies

$$\mu(E) = \sup\{\mu(K) \mid K \subseteq E, \text{for } K \text{ compact}\} = \inf\{\mu(U) \mid E \subseteq U, \text{for } U \text{ open}\},$$

for all $E \in \mathcal{B}$. A *complex Borel measure* is a function $\mu: \mathcal{B} \to \mathbb{C}$ which is countably additive. The *total variation* measure with respect to a complex Borel measure μ is the measure $|\mu|$ given by

$$|\mu|(E) = \sup \sum_i |\mu(E_i)|, \qquad \text{for } E \in \mathcal{E},$$

where the sup is over all countable collections $\{E_i\}$ of disjoint sets of $\mathcal{B}$ such that $\bigcup_i E_i = E$. A *regular* complex Borel measure is a Borel measure on X such that the total variation measure $|\mu|$ is regular. A measure λ is *absolutely continuous* with respect to a measure μ if $\mu(E) = 0$ implies $\lambda(E) = 0$.

Let μ be a Haar measure on a locally compact group G. Under the map

$$\begin{array}{ccc} \{\text{functions}\} & \longrightarrow & \{\text{measures}\} \\ f & \longmapsto & f(g)d\mu(g) \end{array}$$

the group algebra $\mathbb{C}G$ maps to measures ν with finite support, $\ell^(G)$ maps to measures with countable support, and $L^1(G,\mu)$ maps to measures ν which are absolutely continuous with respect to μ.

Let X be a locally compact Hausdorff topological space. Define

$$C_c(X) = \{\text{continuous functions } f\colon X \to \mathbb{C} \text{ with compact support}\}.$$

Then $C_c(X)$ is a normed vector space (not always complete) under the norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}.$$

The completion $C_0(X)$ of $C_c(X)$ with respect to $\|\cdot\|_\infty$ is a Banach space. A *distribution* is a bounded linear functional $\mu\colon C_c(X) \to \mathbb{C}$. The Riesz representation theorem says that with the notation

$$\mu(f) = \int_X f(x)d\mu(x), \qquad \text{for } f \in C_c(X),$$

the regular complex Borel measures on X are exactly the distributions on X. The *norm* $\|\mu\|$ is the norm of μ as a linear functional $\mu\colon C_c(X) \to \mathbb{C}$. Viewing μ as a measure, $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the total variation measure of μ.

The *support* supp μ of a distribution μ is the set of $x \in X$ such that for each neighborhood U of x there is $f \in C_c(X)$ such that $\text{supp}(f) \subseteq U$ and $\mu(f) \neq 0$. Define

$$\mathcal{E}_c(X) = \{\text{distributions } \mu \text{ on } X \text{ with compact support}\}.$$

If $\phi\colon X \to Y$ is a morphism of locally compact spaces then

$$\phi_*\colon \mathcal{E}_c(X) \to \mathcal{E}_c(Y) \quad \text{is given by} \quad (\phi_*\mu)(f) = \mu(f \circ \phi),$$

for $f \in C_c(Y)$.

Let G be a locally compact topological group. Define an involution on distributions by

$$\mu^*(f) = \mu(f^*), \qquad \text{for } f \in C_c(G).$$

The *convolution* of distributions is defined by

$$\int_G f(g)d(\mu_1 * \mu_2)(g) = \int_G \int_G f(g_1g_2)d\mu_1(g_1)d\mu_2(g_2).$$

The left and right actions of G on distributions are given by

$$(L_g\mu)(f) = \mu(L_{g^{-1}}f), \quad \text{and} \quad (R_g\mu)(f) = \mu(R_{g^{-1}}f), \qquad \text{for all } f \in C_c(G).$$

Let X be a smooth manifold. The vector space $C^\infty(X)$ is a topological vector space under a suitable topology. A *compactly supported distribution* on X is a continuous linear functional $\mu: C^\infty(X) \to \mathbb{C}$. Let

$$\mathcal{E}^1(X) = \{\text{continuous linear functionals } \mu: C^\infty(X) \to \mathbb{C}\}$$

and, for a compact subset $K \subseteq X$,

$$\mathcal{E}^1(X, K) = \{\mu \in \mathcal{E}^1(X) \mid \operatorname{supp}(\mu) \subseteq K\}.$$

If $\phi: X \to Y$ is a morphism of smooth manifolds then

$$\phi_*: \mathcal{E}^1(X) \to \mathcal{E}^1(Y) \qquad \text{is given by} \qquad (\phi_*\mu)(f) = \mu(f \circ \phi).$$

Haar measures and the modular function

Let G be a locally compact Hausdorff topological group. A *Haar measure* on G is a linear functional $\mu: C_0(G) \to \mathbb{C}$ such that

(a) (continuity) μ is continuous with respect to the topolgy on $C_0(G)$ given by

$$\|f\|_\infty = \sup\{|f(g)| \mid g \in G\},$$

(b) (positivity) If $f(g) \in \mathbb{R}_{\geq 0}$ for all $g \in G$ then $\mu(f) \in \mathbb{R}_{\geq 0}$,

(c) (left invariance) $\mu(L_g f) = \mu(f)$, for all $g \in G$ and $f \in C_0(G)$.

Theorem 0.12. *(Existence and uniqueness of Haar measure) If G is a locally compact Hausdorff topological group then G has a Haar measure and any two Haar measures are proportional.*

Fix a (left) Haar measure μ on G. A group is *unimodular* if μ is also a right Haar measure on G. The *modular function* is the function $\Delta: G \to \mathbb{R}_{\geq 0}$ given by

$$\mu(f) = \Delta(g)\mu(R_g f), \qquad \text{for all } f \in C_0(G).$$

The fact that the image of Δ is in $\mathbb{R}_{\geq 0}$ is a consequence of the positivity condition in the definition of Haar measure. There are several equivalent ways of defining the modular function

$$\mu(f^*) = \mu(\Delta^{-1} f) \quad \text{or} \quad \int_G f(g) d\mu(gh) = \int_G f(g)\Delta(h) d\mu(g), \quad \text{or} \quad \mu(f) = \mu_R(\Delta f),$$

for all $f \in C_0(G)$, where μ_R is a *right* Haar measure on G. The group G is unimodular exactly when $\Delta = 1$.

Proposition 0.13. *Finite groups, abelian groups, compact groups, semisimple Lie groups, reductive Lie groups, and nilpotent groups are all unimodular.*

Proposition 0.14. *(a) On a Lie group the Haar measure is given by*

$$\mu(f) = \int_G f\omega, \qquad \text{for all } f \in C_0(G),$$

where ω is the unique positive left invariant n form on G. (b) For a Lie group G the modular function is given by

$$\Delta(g) = |\det Ad_g|, \qquad \text{for all } g \in G.$$

Examples

(1) $\mathbb{R}$, under addition. Haar measure is the usual Lebesgue measure dx on $\mathbb{R}$.
(2) $\mathbb{R}_{\geq 0}$, under multiplication. Haar measure is given by $(1/x)dx$.
(3) $GL_n(\mathbb{R})$ has Haar measure $\displaystyle\frac{1}{|\det(x_{ij})|^n} \prod_{i,j=1}^{n} dx_{ij}$.
(4) The group B_n of upper triangular matrices in $GL_n(\mathbb{R})$ has Haar measure $\displaystyle\frac{1}{\prod_{i=1}^n |x_{ii}|^i} \prod_{1\leq i<j\leq n} dx_{ij}$.

This group is not unimodular unless $n = 1$.
(5) A finite group has Haar measure $\mu(f) = \displaystyle\frac{1}{|G|}\sum_{g\in G} f(g)$.

Vector spaces and linear transformations

A *vector space* is a set V with an addition $+: V\times V \to V$ and a scalar multiplication $\mathbb{C}\times V \to V$ such that addition makes V into an abelian group and

$$\begin{aligned} c(v_1+v_2) &= cv_1 + cv_2, \\ c_1(c_2v) &= (c_1c_2)v, \end{aligned} \qquad \text{and} \qquad \begin{aligned} (c_1+c_2)v &= c_1v + c_2v, \\ 1v &= v \end{aligned}$$

for all $c, c_1, c_2 \in \mathbb{C}$ and $v, v_1, v_2 \in V$. A *linear transformation* from a vector space X to a vector space Y is a map $T: X \to Y$ such that $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$, for all $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2 \in V$. The morphisms in the category of vector spaces are linear transformations.

A *topological vector space* is a vector space V with a topology such that addition and scalar multiplication are continuous maps. The morphisms in the category of topological vector spaces are continuous linear transformations. A set $C \subseteq V$ is *convex* if $tx + (1-t)y \in C$, for all $x, y \in C$, $t \in [0, 1]$. A topological vector space V is *locally convex* if it has a basis of nieghborhoods of 0 consisting convex sets.

A *normed linear space* is a vector space V with a norm $\|\cdot\|: V \to \mathbb{R}_{\geq 0}$ such that

(a) $\|x + y\| \leq \|x\| + \|y\|$, for $x, y, \in V$,
(b) $\|\alpha x\| = |\alpha|\|x\|$, for $\alpha \in \mathbb{C}$, $x \in V$,
(c) $\|x\| = 0$ implies $x = 0$.

A linear transformation $T: X \to Y$ between normed vector spaces X and Y is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$. The *norm* of a linear transformation $T: X \to Y$ is

$$\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| \leq 1\}. \tag{0.15}$$

A linear transformation T is *bounded* if $\|T\| < \infty$. If X and Y are normed linear spaces such that points are closed then linear transformation $T: X \to Y$ is continuous if and only if it is bounded (reference??)

A *Banach space* is a normed linear space which is complete with respect to the metric defined by $d(x, y) = \|x - y\|$. A *Hilbert space* is a vector space V with an inner product $\langle,\rangle: V \times V \to \mathbb{C}$ such that for all $c, c_1, c_2 \in \mathbb{C}$ and $v, v_1, v_2, v_3 \in V$,

(a) $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$,

(b) $\langle c_1 v_1 + c_2 v_2, v_3\rangle c_1\langle v_1, v_3\rangle + c_2\langle v_2, v_3\rangle$,
(c) $\langle v, v\rangle = 0$ only if $v = 0$,
(d) V is a Banach space with respect to the norm given by $\|v\|^2 = \langle v, v\rangle$.

If H is a Hilbert space the *adjoint* T^* of a linear transformation $T\colon H \to H$ is the linear transformation defined by

$$\langle Th_1, h_2\rangle = \langle h_1, T^* h_2\rangle, \qquad \text{for all } h_1, h_2 \in H, \tag{0.16}$$

and T is *unitary* if $\langle Tx_1, Tx_2\rangle = \langle x_1, x_2\rangle$ for all $x_1, x_2 \in H$.

Algebras

An *algebra* is a vector space A with an associative multiplication $A \times A$ which satisfies the distributive laws, i.e. such that A is a ring. A *Banach algebra* is a Banach space A with a multiplication such that A is an algebra and

$$\|a_1 a_2\| \le \|a_1\|\|a_2\|, \qquad \text{for all } a_1, a_2 \in A.$$

A **-algebra* is a Banach algebra with an involution $*\colon A \to A$ such that

An element a in a $*$-algebra is *hermitian*, or *self adjoint*, if $a^* = a$. A *C^*-algebra* is a $*$-algebra A such that

$$\|a^* a\| = \|a\|^2, \qquad \text{for all } a \in A.$$

An *idempotented algebra* is an algebra A with a set of idempotents $\mathcal{E}$ such that

(1) For each pair $e_1, e_2 \in \mathcal{E}$ there is an $e_0 \in \mathcal{E}$ such that $e_0 e_1 = e_1 e_0 = e_1$ and $e_0 e_2 = e_2 e_0 = e_2$, and
(2) For each $a \in A$ there is an $e \in \mathcal{E}$ such that $ae = ea = a$. A *von-Neumann algebra* is an algebra A of operators on a Hilbert space H such that
 (a) A is closed under taking adjoints,
 (b) A coincides with its bicommutant.

Examples

1. The algebra $B(H)$ of bounded linear operators on a Hilbert space H with the operator norm (???) and involution given by adjoint (???) is a Banach algebra.

2. Let G be a locally compact Hausdorff topological group G and let μ be a Haar measure on G. The vector space

$$L^2(G, \mu) = \{f\colon G \to \mathbb{C} \mid \|f\|_2 < \infty\}$$

is a Hilbert space under the operations defined in (???).

3. Let V be a vector space. Then $\mathrm{End}(V)$ is an algebra.

Representations

A *representation* of a group G, or *G-module*, is an action of G on a vector space V by automorphisms (invertible linear transformations). A *representation* of an algebra A, or *A-module*, is an action of A on a vector space V by endomorphisms (linear transformations). A morphism $T\colon V_1 \to V_2$ of A-modules is a linear transformation such that $T(av) = aT(v)$, for all $a \in A$ and $v \in V$. An A-module M is *simple*, or *irreducible*, if it has no submodules except 0 and itself.

A *representation* of a topological group G, or G-*module*, is an action of G on a topological vector space V by automorphisms (continuous invertible linear transformations) such that the map

$$\begin{array}{ccc} G \times V & \longrightarrow & V \\ (g, v) & \longmapsto & gv \end{array}$$

is continuous. When dealing with representations of topological groups all submodules are assumed to be closed subspaces.

A *$*$-representation* of a $*$-algebra A is an action of A on a Hilbert space H by bounded operators such that

$$\langle av_1, v_2 \rangle = \langle v_1, a^* v_2 \rangle, \quad \text{for all } v_1, v_2 \in V,\ a \in A.$$

A $*$-representation of A on H is *nondegenerate* if $AV = \{av \mid a \in A, v \in V\}$ is dense in V.

A *unitary representation* of a topological group G, or G-*module*, is an action of G on a Hilbert space V by automorphisms (unitary continuous invertible linear transformations) such that the action $G \times V \longrightarrow V$ is a continuous map.

An *admissible representation* of an idempotented algebra $(A, \mathcal{E})$ is an action of A on a vector space V by linear transformations such that

(a) $V = \bigcup_{e \in \mathcal{E}} eV$,

(b) each eV is finite dimensional.

A representation of an idempotented algebra is *smooth* if it satisfies (a).

Group algebras

(1) Let G be a group. Then $\mathbb{C}G$ is the algebra with basis G and multiplication forced by the multiplication in G and the distributive law. A representation of G on a vector space V extends uniquely to a representation of $\mathbb{C}G$ on V and this induces an equivalence of categories between the representations of G and the representations of $\mathbb{C}G$.

(2) Let G be a locally compact topological group and fix a Haar measure μ on G. Let

$$L^1(G, \mu) = \left\{ f\colon G \to \mathbb{C} \mid \|f\| = \int_G |f(g)| d\mu(g) < \infty \right\}.$$

Then $L^1(G, \mu)$ is a $*$-algebra under the operations defined in (???). Any unitary representation of G on a Hilbert space H extends uniquely to a representation of $L^1(G, \mu)$ on H by the formula

$$fv = \int_G f(g) gv d\mu(g), \qquad f \in L^1(G, \mu), g \in G,$$

and this induces an equivalence of categories between the unitary representations of G and the nondegenerate $*$-representations of $L^1(G, \mu)$.

(3) Let G be a locally compact topological group. and fix a Haar measure μ on G. Let

$$\mathcal{E}_c = \{\text{distributions on } G \text{ with compact support}\}$$

Then $\mathcal{E}_c$ is a ???-algebra under the operations defined in (???). Any representation of the topological group G on a complete locally convex vector space V extends uniquely to a representation of $\mathcal{E}_c$ on V by the formula

$$\mu v = \int_G gv d\mu(g), \qquad f \in \mathcal{E}_c, g \in G,$$

and this induces an equivalence of categories between the representations of G on a complete locally convex vector space V and the representations of $\mathcal{E}_c(G)$ on a complete locally convex vector space V.

(4) Let G be a totally disconnected locally compact unimodular group and fix a Haar measure μ on G. Let

$$C_c(G) = \{\text{locally constant compactly supprted functions } f\colon G \to \mathbb{C}\}.$$

Then $C_c(G)$ is a idempotented algebra with with the operations in (???) and with idempotents given by

$$e_K = \frac{1}{\mu(K)}\chi_K, \qquad \text{for open compact subgroups } K \subseteq G,$$

where χ_K denotes the characteristic function of the subgroup K. Any smooth representation of G extends uniquely to a smooth representation of $C_c(G)$ on V by the formula in (???) and this induces an equivalence of categories between the smooth representations of G and the smooth representations of $C_c(G)$ (see Bump Prop. 3.4.3 and Prop. 3.4.4). This correspondence takes admissible representations for G (see Bump p. 425) to admissible representations for $C_c(G)$.

(5) Let G be a Lie group. Let

$$C_c^\infty(G) = \{\text{compactly supported smooth functions on } G\}.$$

Then $C_c^\infty(G)$ is a ???-algebra under the operations defined in (???). Any representation of the topological group G on a complete locally convex vector space V extends uniquely to a representation of $C_c^\infty(G)$ on V by the formula in (???) and this induces an equivalence of categories between the representations of G on a complete locally convex vector space V and the representations of $C_c^\infty(G)$ on a complete locally convex vector space V.

(6) Let G be a reductive Lie group and let K be a maximal compact subgroup of G. Let

$$\mathcal{E}(G,K)^{\text{fin}} = \{\mu \in \mathcal{E}_c(G) \mid \text{supp}(\mu) \subseteq K \text{ and } \mu \text{ is left and right } K \text{ finite}\}.$$

Then $\mathcal{E}(G,K)^{\text{fin}}$ is a idempotented algebra with with the operations in (???) and with idempotents given by

$$e_K = \frac{1}{\mu(K)}\chi_K, \qquad \text{for open compact subgroups } K \subseteq G,$$

where χ_K denotes the characteristic function of the subgroup K. Any $(\mathfrak{g},K)$-module extends uniquely to a smooth representation of $\mathcal{E}(G,K)^{\text{fin}}$ on V by the formula in (???) and this induces an equivalence of categories between the $(\mathfrak{g},K)$-modules and the smooth representations of $\mathcal{E}(G,K)^{\text{fin}}$ (see Bump Prop. 3.4.8). This correspondence takes admissible modules for G (see Bump p. 280 and p. 193) to admissible modules for $\mathcal{E}(G,K)^{\text{fin}}$. By Knapp and Vogan Cor. 1.7.1

$$\mathcal{E}(G,K)^{\text{fin}} = C(K)^{\text{fin}} \otimes_{U(\mathfrak{k}_\mathbb{C})} U(\mathfrak{g}_\mathbb{C}).$$

(7) Let G be a compact Lie group. Let

$$C(G)^{\text{fin}} = \{f \in C^\infty(G) \mid f \text{ is } G \text{ finite}\}.$$

Then $C(G)^{\text{fin}}$ is an idempotented algebra with idempotents corresponding to the identity on a finite sum of blocks $\bigoplus_\lambda G^\lambda \otimes \overline{G}^\lambda$.

Theorem 0.17. *The category of representations of G in a Hilbert space V and the category of smooth representations of $C(G)^{\mathrm{fin}}$ are equivalent.*

(8) Let $\mathfrak{g}$ be a Lie algebra. The enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is the associative algebra with 1 given by
Generators: $x \in \mathfrak{g}$, and
Relations: $xy - yx = [x, y]$, for all $x \in \mathfrak{g}$.

The functor

$$\begin{array}{rccc} U\colon & \{\text{Lie algebras}\} & \longrightarrow & \{\text{associative algebras}\} \\ & \mathfrak{g} & \longmapsto & U\mathfrak{g} \end{array}$$

is the left adjoint of the functor

$$\begin{array}{rccc} L\colon & \{\text{associative algebras}\} & \longrightarrow & \{\text{Lie algebras}\} \\ & (A, \cdot) & \longmapsto & (A, [,]) \end{array}$$

where $(A, [,])$ is the Lie algebra given by the vector space A with the bracket $[,]\colon A \otimes A \to \mathbb{C}$ defined by

$$[a_1, a_2] = a_1 a_2 - a_2 a_1, \qquad \text{for all } a_1, a_2 \in A.$$

This means that

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, LA) \cong \mathrm{Hom}_{\mathrm{alg}}(U\mathfrak{g}, A), \quad \text{for all associative algebras } A. \tag{0.18}$$

Let $\iota\colon \mathfrak{g} \to U\mathfrak{g}$ be the map given by $\iota(x) = x$. Then (???) is equivalent to the following *universal property* satisfied by $U\mathfrak{g}$:

If $\phi\colon \mathfrak{g} \to A$ is a map from $\mathfrak{g}$ to an associative algebra A such that

$$\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x), \quad \text{for all } x, y, \in \mathfrak{g},$$

then there exists an algebra homomorphism $\tilde{\phi}\colon U\mathfrak{g} \to A$ such that $\tilde{\phi} \circ \iota = \phi$.

A representation of $\mathfrak{g}$ on a vector space V extends uniquely to a representation of $U\mathfrak{g}$ on V and this induces an equivalence of categories between the representations of $\mathfrak{g}$ and the representations of $U\mathfrak{g}$.

Proposition 0.19. *Let G be a Lie group and let $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$ be the complexification of the Lie algebra $\mathfrak{g}_{\mathbb{R}} = Lie(G)$ of G. Let $\mathcal{E}(G, \{1\})$ be the algebra of distributions $\mu\colon C^\infty(G) \to \mathbb{C}$ on G such that $supp(\mu) = \{1\}$. Then*

$$\begin{array}{ccc} U\mathfrak{g} & \longrightarrow & \mathcal{E}(G, \{1\}) \\ x & \longmapsto & \mu_x \end{array} \qquad \text{where} \quad \mu_x(f) = \frac{d}{dt} f(e^{tx})\Big|_{t=0}, \quad \text{for } x \in \mathfrak{g},$$

is an isomorphism of algebras.

Compact groups

Let G be a compact Lie group and let μ be a Haar measure on G. Assume that μ is normalized so that $\mu(G) = 1$. The algebra $C_c(G)$ (under convolution) of continuous complex valued functions on G with compact support is the same as the algebra $C(G)$ of continuous functions on G. The vector space $C(G)$ is a G-module with G-action given by

$$(xf)(g) = f(x^{-1}g), \quad \text{for } x \in G,\ f \in C(G).$$

The group G acts on $C(G)$ in two ways,

$$(L_g f)(x) = f(g^{-1}x), \qquad \text{and} \qquad (R_g f)(x) = f(xg),$$

and these two actions commute with each other.

Suppose that V is a representation of G in a complete locally convex vector space. Let $(,) \colon V \otimes V \to \mathbb{C}$ be an inner product on V and define a new innner product $\langle , \rangle \colon V \otimes V \to \mathbb{C}$ by

$$\langle v_1, v_2 \rangle = \int_G (gv_1, gv_2) d\mu(g), \qquad v_1, v_2 \in V.$$

Under the inner product $\langle , \rangle$ the representation V is unitary. If V is a finite dimensional representation of G,

$$\begin{array}{cccc} V\colon & G & \longrightarrow & M_n(\mathbb{C}) \\ & g & \longmapsto & V(g), \end{array} \qquad \text{then} \qquad \begin{array}{cccc} \bar{V}\colon & G & \longrightarrow & M_n(\mathbb{C}) \\ & g & \longmapsto & \overline{V(g)} = V(g^{-1})^t, \end{array}$$

is another finite dimensional representation of G.

Lemma 0.20. *Every finite dimensional representation of a compact group is unitary and completely decomposable.*

The representation $C(G)$ is an example of an infinite dimensional representation of G which is not unitary.

If V is a representation of G in a complete locally convex normed vector space V then the representation V can be extended to be a representation of the algebra (under convolution) of continuous functions $C(G)$ on G by

$$fv = \int_G f(g) gv d\mu(g), \qquad f \in C(G), v \in V. \tag{0.21}$$

The complete locally convex assumption on V is necessary to define the integral in (???).

If V is a representation of G define

$$V^{\text{fin}} = \{ v \in V \mid \text{the } G\text{-module generated by } v \text{ is finite dimensional}\}.$$

The vector space $C(G)^{\text{rep}}$ of *representative* functions consists of all functions $f \colon G \to \mathbb{C}$ given by

$$f(g) = \langle v, gw \rangle,$$

for some vectors v, w in a finite dimensional representation of G.

Lemma 0.22. *Let G be a compact group. Then $C(G)^{\text{fin}} = C(G)^{\text{rep}}$.*

Proof. Let $f \in C(G)^{\mathrm{rep}}$. Let v, w be vectors in a finite dimensional representation V such that $f(g) = \langle v, gw\rangle$ for all $g \in G$. Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of V and let W be the vector space of linear combinations of the functions $f_j = \langle v_j, gw\rangle$, $1 \le j \le k$. Since v can be wrritten as a linear combinattion of the v_j, the function f can be written as a linear combination of the f_j and so $f \in W$. For each $1 \le i \le k$

$$(xf_i)(g) = \tilde{f}_i(x^{-1}g) = \langle v_i, x^{-1}gw\rangle = \langle xv_i, gw\rangle = \langle \sum_{j=1}^{k} c_j v_j, gw\rangle = \sum_{j=1}^{k} c_j f_j(g)$$

for some constants $c_j \in \mathbb{C}$. So the G-module generated by f is contained in the finite dimensional representation W. So $f \in C(G)^{\mathrm{fin}}$. So $C(G)^{\mathrm{rep}} \subseteq C(G)^{\mathrm{fin}}$.

Let $f \in C(G)^{\mathrm{fin}}$ and let $f_1 = f, f_2, \ldots, f_k$ be an orthonormal basis of the finite dimensional representation W generated by f. Then

$$f(g) = (g^{-1}f_1)(1) = \sum_{j=1}^{k} \langle f_j, g^{-1}f_1\rangle f_j(1), \qquad \text{where } c_j = \langle f_j, g^{-1}f_1\rangle.$$

Define a new finite dimensional representation $\bar{W}$ of G which has orthonormal basis $\{\bar{w}_1, \ldots, \bar{w}_k\}$ and G action given by

$$g\bar{w}_i = \sum_{j=1}^{k} \overline{\langle f_j, g^{-1}f_i\rangle}\bar{w}_j, \qquad 1 \le i \le k.$$

It is straightforward to check that $g_1(g_2\bar{w}) = (g_1g_2)\bar{w}$, for all $g_1, g_2 \in G$. Since $\langle \bar{w}_j, g\bar{w}_i\rangle = \langle f_j, g^{-1}f_i\rangle$,

$$f(g) = \langle \sum_{j=1}^{k} c_j \bar{w}_j, g\bar{w}_1\rangle \quad \text{where } c_j = f_j(1)$$

and so $f \in C(G)^{\mathrm{rep}}$. So $C(G)^{\mathrm{fin}} \subseteq C(G)^{\mathrm{rep}}$. ∎

Theorem 0.23. *(Peter-Weyl) Let G be a compact Lie group. Then*

(a) $C(G)^{\mathrm{rep}}$ is dense in $C(G)$, under the topology defined by the sup *norm.*

(b) V^{fin} is dense in V for all representations V of G.

(c) G is linear, i.e. there is an injective map $i: G \to GL_n(\mathbb{C})$ for some n.

(d) Let $\hat{G}$ be an index set for the finite dimensional representations of G. For each finite dimensional irreducible representation G^λ, $\lambda \in \hat{G}$, fix an orthonormal basis $\{v_i^\lambda \mid 1 \le i \le d_\lambda\}$ of G^λ. Define $M_{ij}^\lambda \in C(G)^{\mathrm{rep}}$ by

$$M_{ij}^\lambda(g) = \langle v_i^\lambda, gv_j^\lambda\rangle, \qquad g \in G.$$

Then

$$\begin{array}{ccc} \bigoplus_{\lambda \in \hat{G}} G^\lambda \otimes G^\lambda & \longrightarrow & C(G)^{\mathrm{rep}} \\ v_i^\lambda \otimes v_j^\lambda & \longmapsto & M_{ij}^\lambda \end{array}$$

is an isomorphism of $G \times G$-modules.

(e) The map

$$\begin{array}{ccc} \bigoplus_{\lambda \in \hat{G}} M_{d_\lambda}(\mathbb{C}) \longrightarrow C(G)^{\mathrm{rep}} & & \\ E_{ij}^\lambda & \longmapsto & M_{ij}^\lambda \end{array}$$

is an isomorphism of algebras.

and (a), (b), (c), (d) and (e) are all equivalent.

Proof. (b) $\Longrightarrow$ (a) is immediate.

(a) $\Longrightarrow$ (b): Note that $C(G)^{\text{fin}}V \subseteq V^{\text{fin}}$. Since $C(G)^{\text{fin}}$ is dense in $C(G)$, the closure of $C(G)^{\text{fin}}V$ contains $C(G)V$. Let $f_1, \ldots, f_2$ be a sequence of functions in $C(G)$ such that $\mu(f_i) = 1$ and the sequence approaches the δ function at 1, i.e. the function δ_1 which has $\text{supp}(\delta_1) = \{1\}$. If $v \in V$ then the sequence $f_1 v, f_2 v, \ldots$ approaches $1v = v$ and so v is in the closure of $C(G)V$. So the closure of $C(G)V$ is V. So V^{fin} is dense in V.

The following method of making this precise is taken more or less from Bröcker and tom Dieck.

An operator $K: C(G) \to C(G)$ is *compact* if, for every bounded $B \subseteq C(G)$, every sequence $(f_n) \subseteq K(B)$ converges in $K(B)$. An operator $K: C(G) \to C(G)$ is *symmetric* if $\langle Kf_1, f_2 \rangle = \langle f_1, Kf_2 \rangle$ for all $f_1, f_2 \in C(G)$.

Proposition 0.24. *See Bröcker-tom Dieck Theorem (2.6) If $K: C(G) \to C(G)$ is a compact symmetric operator then*

(a) $\|K\| = \sup\{\|Kf\| \mid \|f\| \le 1\}$ *or* $-\|K\|$ *is an eigenvalue of* K,

(b) *All eigenspaces of K are finite dimensional,*

(c) $\bigoplus_\lambda C(G)_\lambda$ *is dense in* $C(G)$.

Proof. (b) The reason eigenspaces are finite dimensional: Let $x_1, x_2, \ldots$ be an orthonormal basis. Then $Kx_i = \lambda x_i$. So

$$\|Kx_i - Kx_j\|^2 = |\lambda^2| \|x_1 - x_j\|^2 = 2\|\lambda\|^2$$

and this never goes to zero.

(c) If not then $U^\perp = (\overline{\bigoplus_\lambda C(G)_\lambda})^\perp$ is nonzero. Then $K: U^\perp \to U^\perp$ is a compact symmetric operator. So this operator has a finite dimensional eigenspace. This is a contradiction. So $U^\perp = 0$. So $\bigoplus_\lambda C(G)_\lambda$ is dense in $C(G)$. ∎

Take K to be the operator given by convolution by an approximation ϕ to the δ function. Then Kf is close to f,

$$\begin{aligned} \|Kf - f\|_\infty &= \Big| \int_G (\delta(g) f(xg) - f(g)) d\mu(g) \Big| \le \int_G \epsilon \delta(g) d\mu(g) = \epsilon \\ &= \|\delta(1) - 1\|_\infty \le \epsilon, \end{aligned}$$

and Kf can be approximated by the action of ϕ on finite dimensional subspaces.

The symmetric condition on K translates to

$$\phi(g) = \phi(g^{-1})$$

and the compactness condition translates to

$$\int_G \phi(g) d\mu(g) = 1.$$

Note that

$$\|f\|_2^2 = \int f(g)\overline{f(g)} d\mu(g) \le \int |f(g)\overline{f(g)} d\mu(g) \le \|f\|_\infty^2.$$

So the L^2 and sup norms compare. For norms of operators $\|\delta * f\|_\infty \le \|\delta\|_\infty \|f\|_\infty$.

(c) $\Longrightarrow$ (a): If $\iota: G \to GL_n(\mathbb{C})$ is an injection then the algebra $C(G)^{\rm alg}$ generated (under pointwise multiplication) by the functions ι_{ij} and $\bar\iota_{ij}$, where

$$\iota_{ij}(g) = \iota(g)_{ij}, \quad \text{and} \quad \iota_{ij}(g) = \overline{\iota_{ij}(g)}, \quad \text{for } g \in G,$$

is contained in $C(G)^{\rm fin}$. This subalgebra separates points of G and is closed under pointwise multiplication, and conjugation and so, by the Stone-Weierstrass theorem, is dense in $C(G)$. So $C(G)^{\rm fin}$ is dense in $C(G)$.

(a) $\Longrightarrow$ (c): The elements of $C(G)$ distinguish the points of G and so the functions in $C(G)^{\rm rep}$ distinguish the points of G. For each $g \in G$ fix a function f_g such that $(gf_g)(1) = f_g(g^{-1}) \ne f_g(1)$ and let V_g be the finite dimensional representation of G generated by f_g. By choosing $g_i \notin K_{i-1}$ we can find a sequence $g_1, g_2, \ldots$ of elements of G such that

$$K_1 \supseteq K_2 \supseteq \ldots, \qquad \text{where } K_j = \ker(V_{g_1} \oplus \cdots \oplus V_{g_j}),$$

and $K_i \ne K_{i+1}$. Since each K_i is a closed subgroup of G, and G is compact there is a finite n such that $K_n = \{1\}$. Then $W = V_{g_1} \oplus \cdots V_{g_n}$ is a finite dimensional representation of G with trivial kernel. So there is an injective map from G into $GL(W)$.

(d) By *construction* this an algebra isomorphism. After all the algebra multiplication is designed to extend the $G \times G$ module structure, and this is a $G \times G$ module homomorphism since

$$\begin{aligned}
\big((x\otimes y)(v_i^\lambda \otimes v_j^\lambda)\big)(g) &= \big(\Phi(xv_i^\lambda \otimes yv_j^\lambda)\big)(g) \\
&= \langle xv_i^\lambda \otimes gyv_j^\lambda \\
&= \langle v_i^\lambda \otimes x^{-1}gyv_j^\lambda \\
&= M_{ij}^\lambda(x^{-1}gy) \\
&= (L_x R_y M_{ij}^\lambda)(g).
\end{aligned}$$

∎

Note that

$$\mathrm{Tr}(E_{ij}^\lambda) = \langle v_i^\lambda, v_j^\lambda \rangle = \delta_{ij}.$$

Consider the L^2 norm on $C(G)^{\rm rep}$.

$$\begin{aligned}
\|f\|_2^2 &= \int_G f(g)\overline{f(g)} d\mu(g) \\
&= \int_G f(g) f^*(g^{-1}) d\mu(g) \qquad \text{where } f^*(g) = \overline{f(g^{-1})} \\
&= (f * f^*)(1).
\end{aligned}$$

More generally, $\langle f_1, f_2\rangle_2 = (f_1 * f_2)(1)$. Now

$$\begin{array}{cccc}
\tau: & C(G)^{\rm rep} & \longrightarrow & \mathbb{C} \\
f & \longmapsto & & f(1)
\end{array}$$

is a trace on $C(G)^{\text{rep}}$, i.e. $\tau(f_1 * f_2) = \tau(f_2 * f_1)$ for all $f_1, f_2 \in C(G)^{\text{rep}}$. In fact thsi is trace of the action of $C(G)^{\text{rep}}$ on itself:

$$\begin{aligned}\tau(f) &= \int_G f(g)gh\big|_h d\mu(g) \\ &= \int_G f(g)\delta_{g1} d\mu(g) \\ &= \int_G f(1)d\mu(g) = f(1)\mu(G) = f(1).\end{aligned}$$

Now consider the action of $\bigoplus_\lambda M_{d_\lambda}(\mathbb{C})$ on itself. Then, if $f = (\hat{f}^\lambda)$ then

$$\tau(f) = \sum_{\lambda\in\hat{G}} d_\lambda \mathrm{Tr}(f^\lambda).$$

So

$$\|f\|_2^2 = (f * f^*)(1) = \tau(f * f^*) = \tau(\hat{f}^\lambda(\hat{f}^\lambda)^t) = \sum_{\lambda\in\hat{G}} d_\lambda \mathrm{Tr}(\hat{f}^\lambda(\hat{f}^\lambda)^t).$$

Note that $\mathrm{Tr}(\mathrm{Id}_\lambda) = d_\lambda$ and $\tau(\mathrm{Id}_\lambda) =$???.

Fourier analysis for compact groups

A function $f: G \to \mathbb{C}$ is

(a) *representative* if there is a finite dimensional representation V of G and vectors $v, w \in V$ such that $f(g) = \langle v, gw\rangle$ for all $g \in G$.

(b) *square integrable* if

$$\|f\|_2^2 = \int_G f(g)\overline{f(g)}d\mu(g) < \infty.$$

(c) *smooth* if all derivatives of f exist.

(d) *real analytic* if f has a power series expansion at every point.

$$\begin{aligned}C(G)^{\text{rep}} &= \{\text{representative functions } f: G \to \mathbb{C}\}, \\ L^2(G) &= \{\text{square integrable functions } f: G \to \mathbb{C}\}, \\ C^\infty(G) &= \{\text{smooth functions } f: G \to \mathbb{C}\}, \\ C^\omega(G) &= \{\text{real analytic functions } f: G \to \mathbb{C}\},\end{aligned}$$

We have a map

$$\prod_{\lambda\in\hat{G}} M_{d_\lambda}(\mathbb{C}) \longrightarrow \text{functions } f: G \to \mathbb{C}.$$

The set $\hat{G}$ has a norm $\|\cdot\|: \hat{G} \to \mathbb{R}_{\geq 0}$. For $(\hat{f}^\lambda) \in \prod_{\lambda\in\hat{G}} M_{d_\lambda}(\mathbb{C})$ define

(a) $(\hat{f}^\lambda)$ is *finite* if all but a finite number of the blocks $\hat{f}^\lambda$ in $(\hat{f}^\lambda)$ are 0,

(b) $(\hat{f}^\lambda)$ is *square summable* if

$$\sum_{\lambda\in\hat{G}} \frac{1}{d_\lambda}\|f^\lambda\|^2 < \infty.$$

(c) $(\hat{f}^\lambda)$ is *rapidly decreasing* if, for all $k \in \mathbb{Z}_{>0}$, $\{\|\lambda\|^k\|\hat{f}^\lambda\| \mid \lambda in\hat{G}\}$ is bounded,

(d) $(\hat{f}^\lambda)$ is *exponentially decreasing* if, for some $K \in \mathbb{R}_{>1}$, $\{K^{\|\lambda\|}\|\hat{f}^\lambda\| \mid \lambda \in hatG\}$ is bounded.

Under the map

$$\begin{array}{ccc} \{\text{functions } f: G \to \mathbb{C}\} & \longrightarrow & \prod_{\lambda \in \hat{G}} M_{d_\lambda}(\mathbb{C}), \\ C(G)^{\text{rep}} & \longmapsto & \{\text{finite } (\hat{f}^\lambda)\} \\ L^2(G,\mu) & \longmapsto & \{\text{square summable } (\hat{f}^\lambda)\} \\ C^\infty(G) & \longmapsto & \{\text{rapidly decreasing } (\hat{f}^\lambda)\} \\ C^\omega(G) & \longmapsto & \{\text{exponentially decreasing } (\hat{f}^\lambda)\} \end{array}$$

The space $C(g)^{\text{rep}}$ is dense in $C(G)$ and $C(G) \subseteq L^2(G)$. In fact the sup norm on $C(G)$ is related to the L^2 norm on $L^(G)$ and $C(G)$ is dense in $L^2(G)$.

Abelian Lie groups

Theorem 0.25.

(a) *If G is a connected abelian Lie group then*

$$G \cong (S^1)^k \times \mathbb{R}^{n-k},$$

for some $n \in \mathbb{Z}_{>0}$, $0 \le k \le n$.

(b) *If G is a compact abelian Lie group then*

$$G \cong (S^1)^k \times \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_\ell\mathbb{Z},$$

for some $k \in \mathbb{Z}_{\ge 0}$, $m_1, \ldots, m_\ell \in \mathbb{Z}_{>0}$.

Proof. (Sketch) (a)

$$0 \longrightarrow K \longrightarrow \mathfrak{g} \stackrel{\exp}{\longrightarrow} G \longrightarrow 0, \quad \text{where} \quad K = \ker(\exp).$$

The map exp is surjective since the image contains a set of generators of G. The group K is discrete since exp is a local bijection. So $K \cong \mathbb{Z}^k$ since it is a discrete subgroup of a vector space. So

$$G \cong \mathfrak{g}/K \cong \mathbb{R}^n/\mathbb{Z}^k \cong (\mathbb{R}^k/\mathbb{Z}^k) \times \mathbb{R}^{n-k}.$$

(b) Let $T = G^0$. Then $0 \to T \to G \to G/T \to 0$ and G/T is discrete and compact since T is open in G. Thus, by part (a), $T \cong (S^1)^k$, and G/T is finite. So

$$G \cong (S^1)^k \times (\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_\ell\mathbb{Z}).$$

Proposition 0.26.

(a) *The finite dimensional irreducible representations of $\mathbb{Z}/r\mathbb{Z}$ are*

$$\begin{array}{ccc} X^\lambda: & \mathbb{Z}/r\mathbb{Z} & \longrightarrow & \mathbb{C}^* \\ e^{2\pi i k/r} & \longmapsto & e^{2\pi i k\lambda/r} \end{array}, \qquad 0 \le \lambda \le r-1.$$

(b) *The finite dimensional irreducible representations of S^1 are*

$$\begin{array}{ccc} X^\lambda: & \mathbb{Z}/r\mathbb{Z} & \longrightarrow & \mathbb{C}^* \\ e^{2\pi i \beta} & \longmapsto & e^{2\pi i \beta\lambda} \end{array}, \qquad \lambda \in \mathbb{Z}.$$

(c) *The finite dimensional irreducible representations of $\mathbb{Z}$ are*

$$\begin{array}{ccc} z: & \mathbb{Z} & \longrightarrow & \mathbb{C}^* \\ r & \longmapsto & z^r = e^{2\pi i \lambda r} \end{array}, \qquad z \in \mathbb{C}^*, \lambda \in \mathbb{C}.$$

(d) *The finite dimensional irreducible representations of $\mathbb{R}$ are*

$$\begin{array}{ccc} z: & \mathbb{R} & \longrightarrow & \mathbb{C}^* \\ r & \longmapsto & z^r = e^{2\pi i \lambda r} \end{array}, \qquad z \in \mathbb{C}^*, \lambda \in \mathbb{C}.$$

Weights and roots

Let G be a compact connected group. A *maximal torus* of G is a maximal connected subgroup of G isomorphic to $(S^1)^k$ for some positive integer k.

Fix a maximal torus T in G. The group T is a maximal connected abelian subgroup of G. The *Weyl group* is

$$W = N_G(T)/T, \qquad \text{where } N_G(T) = \{g \in G \mid gTg^{-1} = T\}.$$

The Weyl group W acts on T by conjugation. The map

$$\begin{array}{ccc} G/T \times T & \xrightarrow{\phi} & G \\ (gT, t) & \longmapsto & (gtg^{-1}) \end{array}$$

is surjective and $\mathrm{Card}(\phi^{-1}(g)) = |W|$ for any $g \in G$. It follows from this that

(a) Every element $g \in G$ is in some maximal torus.

(b) Any two maximal tori in G are conjugate.

Thus, maximal tori exist, are unique up to conjugacy, and cover the group G.

Let P be an index set for the irreducible representations of T. Since the irreducible representations of S^1 are indexed by $\mathbb{Z}$, $P \cong \mathbb{Z}^k$. The set P is called the *weight lattice* of G.

$$\text{If } \lambda \in P \qquad \text{then} \qquad X^\lambda : T \to \mathbb{C}^*,$$

denotes the corresponding irreducible representation of T. The W-action on T induces a W-action on P via

$$X^{w\lambda}(t) = X^\lambda(w^{-1}t), \qquad \text{for all } t \in T.$$

A representation V of G is a representation of T, by restriction, and, as a T-module,

$$V = \bigoplus_{\lambda \in P} V_\lambda, \qquad \text{where} \quad V_\lambda = \{v \in V \mid tv = X^\lambda(t)v \text{ for all } t \in T.\}$$

The vector space V_λ is the X^λ isotypic component of the T-module V. The W-action on T gives

$$\dim(V_\lambda) = \dim(V_{w\lambda}), \qquad \text{for all } w \in W \text{ and } \lambda \in P.$$

The vector space V_λ is the λ-*weight space* of V. A *weight vector* of *weight* λ in V is a vector v in V_λ.

Let G be a compact connected Lie group and let $\mathfrak{u} = \mathrm{Lie}(G)$. The group G acts on $\mathfrak{u}$ by the adjoint representation. Extend the adjoint representation to be a representation of G on the complex vector space

$$\mathfrak{g}_\mathbb{C} = \mathfrak{u} \oplus i\mathfrak{u} = \mathbb{C} \otimes \mathbb{R}\mathfrak{u}.$$

By ???, this representation extends to a representation of the complex algebraic group $G_\mathbb{C}$ which is the complexification of G. Since G is compact, the adjoint representation of $G_\mathbb{C}$ on $\mathfrak{g}_\mathbb{C}$, and thus the adjoint representation of $\mathfrak{g}_\mathbb{C}$ on itself, is completely decomposable. This shows that $\mathfrak{g}_\mathbb{C}$ is a complex *semisimple* Lie algebra.

The adjoint representation $\mathfrak{g}_\mathbb{C}$ of G has a weight decomposition

$$\mathfrak{g}_\mathbb{C} = \bigoplus_{\alpha \in P} \mathfrak{g}_\alpha,$$

and the *root system* of G is the set

$$R = \{\alpha \in P \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$$

of nonzero weights of the adjoint representation. The *roots* are the elements of R. Set $\mathfrak{h} = \mathfrak{g}_0$. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

is the decomposition of $\mathfrak{g}_{\mathbb{C}}$ into the *Cartan subalgebra* $\mathfrak{h}$ and the *root spaces* $\mathfrak{g}_\alpha$. (Note that the usual notation is $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$, $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \oplus i\mathfrak{h}$, where $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{g}$, i.e. a maximal abelian subspace of $\mathfrak{g}$. Also $\mathfrak{g}_0 = \mathfrak{h}_{\mathbb{C}}$ since $\mathfrak{h}$ is maximal abelian in $\mathfrak{g}$. Also $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of the maximal torus T of G, and the maximal abelian subalgebra in $\mathfrak{g}$. Don't forget to think of

$$X^{\cdot}\colon \begin{array}{ccc} T & \longrightarrow & \mathbb{C}^* \\ t & \longmapsto & X^\lambda(t) \\ e^h & \longmapsto & e^{\lambda(h)} \end{array} \qquad \lambda\colon \begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathbb{C} \\ h & \longmapsto & \lambda(h) \end{array}$$

)

Proposition 0.27.

(?) *The Weyl group W is generated by s_α, $\alpha \in R$. The action of W on $\mathfrak{h}^*$ is generated by the transformations*

$$s_\alpha\colon \begin{array}{ccc} \mathfrak{h}^* & \longrightarrow & \mathfrak{h}^* \\ \lambda & \longmapsto & \lambda - \langle \lambda, \alpha^\vee \alpha \end{array} \qquad \text{where} \qquad \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle},$$

and $\langle , \rangle\colon \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{R}$ is a nondegenerate symmtric bilinear form.

(1) *If α is a root then $-\alpha$ is a root and $\pm\alpha$ are the only multiples of α which are root. (The thing that makes this work is that the root spaces are pure imaginary.)*

(2) *If α is a root then* $\dim(\mathfrak{g}_\alpha) = 1$.

(3) *The only connected compact Lie groups with* $\dim(T) = 1$ *are* $SO_3(\mathbb{R})$ *and the two fold simply connected cover of* $SO_3(\mathbb{R})$.

Proof. (1) Suppose that α is a root and that $x \in \mathfrak{g}_\alpha$.

$$X^\alpha\colon \begin{array}{ccc} T & \longrightarrow & \mathbb{C}^* \\ e^h & \longmapsto & e^{\alpha(h)} \end{array} \qquad \text{and} \qquad X^{-\alpha}\colon \begin{array}{ccc} T & \longrightarrow & \mathbb{C}^* \\ e^h & \longmapsto & \overline{e^{\alpha(h)}} = e^{-\alpha(h)} \end{array}$$

since $\alpha(h) \in i\mathbb{R}$ for $h \in \mathfrak{t}$. Then, for all $h \in \mathfrak{t}$,

$$[h\bar{x}] = [\bar{h}, \bar{x}] = \overline{[h, x]} = \overline{\alpha(h)}\bar{x} = -\alpha(h)\bar{x},$$

and so $\bar{x} \in \mathfrak{g}_{-\alpha}$. Thus $\mathfrak{g}_{-\alpha} \neq 0$ and $-\alpha$ is a root. Note that $[x, \bar{x}] \in \mathfrak{h}$ since it has weight 0.

(2) Consider $X^\alpha\colon T \to \mathbb{C}^*$. Then $T_\alpha = \ker X^\alpha$ is closed in T and is of codimension 1. Let T_α° be the connected component of the identity in T_α and let $Z_\alpha = Z_G(T_\alpha^\circ)$ be the centralizer of T_α° in U (this is connected). Then

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(Z_\alpha) = \mathfrak{t} \oplus i\mathfrak{t} \oplus \left(\oplus_{\substack{h \in T_\alpha \\ \beta(h)=1}} \mathfrak{g}_\beta \right) = \mathfrak{h} \oplus \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{k\alpha}.$$

Now

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & Z_\alpha/T_\alpha^\circ \\ \cup\!| & & \cup\!| \\ T & \longrightarrow & T/T_\alpha^\circ \end{array}$$

So T/T_α° is a maximal torus of Z_α/T_α° and $\dim T/T_\alpha^\circ = 1$. Then

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}(Z_\alpha) = \mathfrak{h}_\alpha \oplus \mathbb{C}H_\alpha \oplus \left(\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{k\alpha} \right).$$

If $X_\alpha \in \mathfrak{g}_\alpha$ then $[X_\alpha, X_{-\alpha}] = \lambda H_\alpha$ and $\lambda \neq 0$ since $\mathbb{C}H$ is maximal abelian in

$$\mathrm{Lie}(Z_\alpha/T_\alpha^\circ) = \mathbb{C}H \oplus \left(\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{k\alpha} \right).$$

Now consider the action of H_α on

$$\mathbb{C}H \oplus \left(\bigoplus_{k\in\mathbb{Z}_{>0}} \mathfrak{g}_{k\alpha} \right) \oplus \mathbb{C}X_\alpha.$$

Then

$$\mathrm{Tr}(H) = \frac{1}{\lambda}\mathrm{Tr}([X_\alpha, X_{-\alpha}]) = \frac{1}{\lambda}\mathrm{Tr}\big(\mathrm{ad}_{X_\alpha}\mathrm{ad}_{X_{-\alpha}} - \mathrm{ad}_{X_{-\alpha}}\mathrm{ad}_{X_\alpha}\big) = 0.$$

But this implies

$$0 = 0 + \sum_{k\in\mathbb{Z}_{>0}} \dim(\mathfrak{g}_{k\alpha}) k\alpha(H_\alpha) - \alpha(H_\alpha).$$

So $\mathfrak{g}_{k\alpha} = 0$ for $k > 1$ and $\mathfrak{g}_\alpha = \mathbb{C}X_\alpha$. So $\mathrm{span}\{X_\alpha, X_{-\alpha}, H_\alpha\}$ is a 3 dimensional subalgebra of $\mathfrak{g}$.

If U is a compact connected Lie group such that $\dim T = 1$ then U has Lie algebra

$$\mathfrak{g} = \mathrm{span}\{X_\alpha, X_{-\alpha}, H_\alpha\} = \mathfrak{u} \oplus i\mathfrak{u}.$$

Then the Weyl group of U is $\{1, s_\alpha\} \cong S_2$ where s_α comes from conjugation by an element of Z_α and so s_α leaves T_α fixed.

So the Weyl group of G contains all the s_α, $\alpha \in R$. ∎

Example. There are only two compact connected groups of dimension 3,

$$SO(3) \qquad \text{and} \qquad \mathrm{Spin}(3).$$

Proof. G acts on $\mathfrak{g}$ and this gives an imbedding $\mathrm{Ad}\colon G \to SO(\mathfrak{g})$ (with respect to an Ad invariant form on $\mathfrak{g}$). This is an immersion since everything is connected. So G is a cover of $SO(3)$. ∎

Weyl's integral formula

Theorem 0.28. *Let G be a compact connected Lie group. Let T be a maximal torus of G and let W be the Weyl group. Let R be the set of roots. Then*

$$|W|\int_G f(x)dx = \int_T \prod_{\alpha\in R}(X^\alpha(t)-1)\int_G (f(gtg^{-1})dgdt.$$

Proof. First note that the map $G/T\times T\to G$ given by $(gT,t)\mapsto gt$, can be used to define a (left) G invariant measure on G/T so that

$$\int_G f(g)dg = \int_{G/T\times T} f(gt)dtd(gT),$$

and thus, for $y\in T$,

$$\begin{aligned}\int_G f(gyg^{-1})dg &= \int_{G/T\times T} f(gtyt^{-1}g^{-1})dtd(gT)\\ &= \int_{G/T\times T} f(gyg^{-1})dtd(gT) = \int_{G/T} f(gyg^{-1})d(gT).\end{aligned} \tag{a}$$

Then the map $\phi: G/T\times T\to G$ given by $(gT,t)\mapsto gtg^{-1}$ yields

$$|W|\int_G f(g)dg = \int_{G/T\times T} f(gtg^{-1})J_{(gT,t)}dtd(gT), \tag{b}$$

where $J_{(gT,t)}$ is the determinant of the differential at (gT,t) of the map ϕ. By translation, $J_{(gT,t)}$ is the same as the determinant of the differential at the identity, (T,e), of the map $L_{gt^{-1}g^{-1}}\circ\phi\circ L_{g,t}$,

$$\begin{array}{ccccccc} G/T\times T & \longrightarrow & G/T\times T & \longrightarrow & G & \longrightarrow & G\\ (xT,y) & \longmapsto & (gxT,ty) & \longmapsto & (gx)ty(gx)^{-1} & \longmapsto & (gt^{-1}g^{-1})(gx)ty(gx)^{-1}.\end{array}$$

Since $(gt^{-1}g^{-1})(gx)ty(gx)^{-1} = gt^{-1}xtyx^{-1}g^{-1}$ this differential is

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h}\oplus\mathfrak{h} & \longmapsto & \mathfrak{g}\\ (X,Y) & \longmapsto & \mathrm{Ad}_g(\mathrm{Ad}_{t^{-1}}(X)+Y-X).\end{array}$$

So $J_{(gT,t)}$ is the determinant of the linear transformation of $\mathfrak{g}$ given by

$$\mathrm{Ad}_{\mathfrak{g}}(g)\begin{pmatrix}\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(t^{-1})-\mathrm{id}_{\mathfrak{g}/\mathfrak{h}} & 0\\ 0 & \mathrm{id}_{\mathfrak{h}}\end{pmatrix},$$

where the second factor is a block 2×2 matrix with respect to the decomposition $\mathfrak{g}/\mathfrak{h}\oplus\mathfrak{h}$ and $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$ is the adjoint action of T restricted to the subspace $\mathfrak{g}/\mathfrak{h}$ in $\mathfrak{g}$. The element t^{-1} acts on the root space $\mathfrak{g}_\alpha$ by the value $X^\alpha(t^{-1})$ where $X^\alpha: T\to\mathbb{C}^*$ is the character of T associated to the root α. Since G is unimodular $\det(\mathrm{Ad}_g)=1$, and since $\mathfrak{g}/\mathfrak{h}=\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$,

$$J_{(gT,t)} = \prod_{\alpha\in R}(X^\alpha(t^{-1})-1) = \prod_{\alpha\in R}(X^\alpha(t)-1), \tag{c}$$

where the last equality follows from the fact that if α is a root then $-\alpha$ is also a root. The theorem follows by combining (a), (b) and (c). ∎

It follows from this theorem that, if χ and η are class functions on G then

$$\begin{aligned}\int_G \chi(g)\overline{\eta(g)}dg &= \frac{1}{|W|}\int_T \prod_{\alpha\in R}(X^\alpha(t)-1)\int_G \chi(gtg^{-1})\overline{\eta(gtg^{-1})}dg\,dt \\ &= \frac{1}{|W|}\int_T \prod_{\alpha>0}(X^\alpha(t)-1)(X^{-\alpha}(t)-1)\chi(t)\overline{\eta(t)}dt \\ &= \frac{1}{|W|}\int_T \prod_{\alpha>0}(X^{\alpha/2}(t)-X^{-\alpha/2}(t))(X^{-\alpha/2}(t)-X^{\alpha/2}(t))\chi(t)\overline{\eta(t)}dt \\ &= \frac{1}{|W|}\int_T \prod_{\alpha>0}(a_\rho\chi)(t)\overline{(a_\rho\eta)(t)}dt\,.\end{aligned}$$

Weyl's character formula

The adjoint representation $\mathfrak{g}$ is a unitary representation of G. So the Weyl group W acts on $\mathfrak{h}$ by unitary operators. So W acts on $\mathfrak{t}$ by orthogonal matrices. Identify $\mathfrak{t}$ and $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t},\mathbb{R}) = \{\alpha\colon \mathfrak{t}\to\mathbb{R}\}$ with the inner product,

$$\begin{array}{ccc}\mathfrak{t} & \xrightarrow{\sim} & \mathfrak{t}^* \\ \alpha & \longmapsto & \langle\alpha,\cdot\rangle.\end{array}$$

For a root α define

$$\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle} \qquad\text{and}\qquad H_\alpha = \{x\in\mathfrak{t}\mid \alpha(x)=0\}.$$

Then, the reflection s_α in the hyperplane H_α, which comes from $Z_\alpha = Z_G(T_\alpha^\circ)/T_\alpha^\circ$, is

$$\begin{array}{cccc} s_\alpha\colon & \mathfrak{t} & \longrightarrow & \mathfrak{t} \\ & \lambda & \mapsto & \lambda - \langle\lambda,\alpha^\vee\rangle\alpha.\end{array}$$

$PICTUREOFHYPERPLANEANDREFLECTION.$

So

(a) W acts on $\mathfrak{t}$, and

(b) $\mathfrak{t} - \bigcup_{\alpha\in R} H_\alpha = \mathbb{R}^n\backslash\left(\bigcup_{\alpha\in R} H_\alpha\right)$ is a union of chambers (these are the connected components).

$PICTUREOFCHAMBERSANDWEIGHTLATTICE$

The Weyl group W permutes these chambers and if we fix a choice of a chamber C then we can identify the chambers are wC, $w\in C$. (See Bröcker-tom Dieck V (2.3iv) and the *Claim* at the bottom of p. 193.

$PICTUREOFCHAMBERSLABELEDBYwC$

Let

$$\begin{aligned}R(T) &= \text{representation ring of } T \\ &= \text{Grothendieck ring of representations of } G\text{, and} \\ R(G) &= \text{representation ring of } G.\end{aligned}$$

This means that $R(G) = \text{span-}\{[G^\lambda] \mid \lambda \in \hat{G}\}$ with

(a) addition given by $[G^\lambda] + [G^\mu] = [G^\lambda \oplus G^\mu]$, and

(b) multiplication given by $[G^\lambda][G^\mu] = [G^\lambda \otimes G^\mu]$.

Thus, in $R(G)$ it makes sense to write

$$\sum_{\lambda \in \hat{G}} m_\lambda [G^\lambda] \qquad \text{instead of} \qquad \bigoplus_{\lambda \in \hat{G}} (G^\lambda)^{\oplus m_\lambda}.$$

Define

$$\mathbb{C}P = \text{span-}\{e^\lambda \mid \lambda \in P\} \quad \text{with multiplication} \quad e^\lambda e^\mu = e^{\lambda+\mu},$$

for $\lambda, \mu \in P$. Then

$$\mathbb{C}P \cong R(T), \qquad \text{since} \qquad R(T) = \text{span-}\{[X^\lambda] \mid \lambda \in P\}.$$

The action of W on $R(T)$ (see (???)) induces an action of W on $\mathbb{C}P$ given by

$$we^\lambda = e^{w\lambda}, \qquad \text{for } w \in W,\, \lambda \in P.$$

Note that

$$\varepsilon(w) = \det_{\mathfrak{h}}(w) = \pm 1$$

since the action of w on $\mathfrak{h}$ is by an orthogonal matrix. The vector spaces of *symmetric* and *alternating* functions are

$$\mathbb{C}[P]^W = \{f \in \mathbb{C}P \mid wf = f \text{ for all } w \in W\}, \qquad \text{and}$$
$$\mathcal{A} = \{f \in \mathbb{C}P \mid wf == \varepsilon(w)f \text{ for all } w \in W\},$$

respectively. Note that $\mathbb{C}[P]^W$ is a ring but $\mathcal{A}$ is only a vector space.

Define

$$P^+ = P \cap \overline{C} \qquad \text{and} P^+ + = P \cap C.$$

The set P^+ is the set of *dominant weights*. Every W-orbit on P contains a unique element of P^+ and so the set of *monomial symmetric functions*

$$m_\lambda = \sum_{\gamma \in W\lambda} e^\gamma, \qquad \lambda \in P^+,$$

forms a basis of $\mathbb{C}[P]^W$. Define

$$a_\mu = \sum_{w \in W} \varepsilon(w) e^{w\mu},$$

for $\mu \in P$. Then

(a) $wa_\mu = \varepsilon(w)a_\mu$, for all $w \in W$ and all $\mu \in P$,

(b) $a_\mu = 0$, if $\mu \in H_\alpha$ for some α, and

(c) $\{a_\mu \mid \mu \in P^+ +\}$ is a basis of $\mathcal{A}$.

The *fundamental weights* $\omega_1, \ldots, \omega_n$ in $\mathfrak{t}$ are defined by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij},$$

where H_{α_j} are the walls of C. Write

$$\alpha > 0 \qquad \text{if } \langle \lambda, \alpha \rangle > 0 \text{ for all } \lambda \in C.$$

Then

$$\begin{aligned}\rho &= \sum_{i=1}^{n} \omega_i \\ &= \frac{1}{2} \sum_{\alpha > 0} \alpha,\end{aligned}$$

is the element of $\mathfrak{t}$ defined by

$$\langle \rho, \alpha_i^\vee \rangle = 1, \qquad \text{for all } \alpha_1, \ldots, \alpha_n.$$

Lemma 0.29. *The map*

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array}$$

is a bijection, and

$$\begin{array}{ccc} \mathbb{C}[P]^W & \longrightarrow & \mathcal{A} \\ f & \longmapsto & a_\rho f \end{array}$$

is a vector space isomorphism.

Proof. Since

$$w(a_\rho f) = (w a_\rho)(w f) = \varepsilon(w) a_\rho f,$$

the second map is well defined. Let

$$g = \sum_{\lambda \in P} g_\lambda e^\lambda \quad \in \mathcal{A}.$$

Then, for a positive root α,

$$-g = s_\alpha g = \sum_{\lambda \in P} g_\lambda e^{s_\alpha \lambda},$$

and so

$$g = \sum_{\substack{\lambda \\ \langle \lambda, \alpha \rangle > 0}} g_\lambda (e^\lambda - s^{s_\alpha \lambda}).$$

Since

$$e^\lambda - e^{s_\alpha \lambda} = (e^{\lambda - \alpha} + \cdots + e^{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha})(e^\alpha - 1),$$

the element g is divsible by $e^\alpha - 1$. Thus, since all the factors in the product are coprime in $\mathbb{C}P$, g is divisible by

$$\prod_{\alpha > 0} (e^\alpha - 1) = e^\rho \prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho a_\rho,$$

where the last equality follows from the fact that a_ρ is divisible by the product $\prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})$ and these two expressions have the same top monomial, e^ρ. Since $g \in \mathcal{A}$ is divisible by a_ρ the map $\mathbb{C}P \to \mathcal{A}$ is invertible. ∎

Define

$$\chi^\lambda = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad \text{for } \lambda \in P^+,$$

so that the $\{\chi^\lambda \mid \lambda \in P^+\}$ are the basis of $\mathbb{C}[P]^W$ obtained by taking the inverse image of the basis $\{a_{\lambda+\rho} \mid \lambda \in P^+\}$ of $\mathcal{A}$. Extend these functions to all of U by setting

$$\chi^\lambda(gtg^{-1}) = \chi^\lambda(t), \qquad \text{for all } g \in U.$$

Since $\int_T X^\lambda(t) X^\mu(t) dt = \delta_{\lambda\mu}$, for $\lambda, \mu \in P$,

$$\int_T a_{\lambda+\rho}(t)\overline{a_{\mu+\rho}(t)}dt = \delta_{\lambda\mu}|W|,$$

and thus, by (???),

$$\delta_{\lambda\mu} = \int_G \chi^\lambda(g)\overline{\chi^\mu(g)}dg, \qquad \text{for all } \lambda, \mu \in P^+.$$

Thus the χ^λ, $\lambda \in P^+$ are an orthonormal basis of the set of class functions in $C(G)^{\mathrm{rep}}$. If U^λ is an irreducible rpresentation of U then

$$\mathrm{Tr}_{U^\lambda}(g) = \sum_{i=1}^{d} M_{ii}^\lambda(g), \qquad \text{where} \qquad M_{ij}^\lambda = \langle v_i^\lambda, g v_j^\lambda \rangle,$$

for an orthonormal basis $v_1^\lambda, \ldots, v_n^\lambda$ of U^λ. Then

$$\int_G \mathrm{Tr}_{U^\lambda}(g)\overline{\mathrm{Tr}_{U^\mu}(g)}dg = \delta_{\lambda\mu},$$

and so the functions Tr_{U^λ} are another orthonormal basis of the set of class functions in $C(G)^{\mathrm{rep}}$. It follows that $\chi^\lambda = \pm\mathrm{Tr}_{U^\lambda}$.

It only remains to check that the sign is positive to show that the χ^λ are the irreducible characters of U. This follows from the following computation.

$$\begin{aligned}
\chi^\lambda(1) &= \lim_{t\to 0}\chi^\lambda(e^{t\rho}) \\
&= \lim_{t\to 0}\frac{\sum_{w\in W}\varepsilon(w)X^{w(\lambda+\rho)}(e^{t\rho})}{\sum_{w\in W}\varepsilon(w)X^{w\rho}(e^{t\rho})} \\
&= \lim_{t\to 0}\frac{\sum_{w\in W}\varepsilon(w)e^{\langle w(\lambda+\rho),t\rho\rangle}}{\sum_{w\in W}\varepsilon(w)e^{\langle w\rho,t\rho\rangle}} \\
&= \lim_{t\to 0}\frac{\sum_{w\in W}\varepsilon(w)e^{t\langle\lambda+\rho,w^{-1}\rho\rangle}}{\sum_{w\in W}\varepsilon(w)e^{t\langle\rho,w^{-1}\rho\rangle}} \\
&= \lim_{t\to 0}\frac{a_\rho(e^{t(\lambda+\rho)})}{a_\rho(e^{t\rho})} \\
&= \lim_{t\to 0}\frac{\prod_{\alpha>0}(X^{\alpha/2}-X^{-\alpha/2})(e^{t(\lambda+\rho)})}{\prod_{\alpha>0}(X^{\alpha/2}-X^{-\alpha/2})(e^{t\rho})} \\
&= \lim_{t\to 0}\frac{\prod_{\alpha>0}\left(e^{t\langle\lambda+\rho,\alpha/2\rangle}-e^{-t\langle\lambda+\rho,\alpha/2\rangle}\right)}{\prod_{\alpha>0}\left(e^{t\langle\rho,\alpha/2\rangle}-e^{-t\langle\rho,\alpha/2\rangle}\right)} \\
&= \lim_{t\to 0}\prod_{\alpha>0}\frac{\sinh(t\langle\lambda+\rho,\alpha/2\rangle)}{\sinh(t\langle\rho,\alpha/2\rangle)} \\
&= \prod_{\alpha>0}\frac{\langle\lambda+\rho,\alpha/2\rangle}{\langle\rho,\alpha/2\rangle} \\
&= \prod_{\alpha>0}\frac{\langle\lambda+\rho,\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle}.
\end{aligned}$$

Theorem 0.30. *Let U be a compact connected Lie group and let T be a maximal torus and L the corresponding lattice.*

(a) The irreducible representations of U are indexed by dominant integral weights $\lambda\in L^+$ underr the corresopndence

$$\begin{array}{ccc}
\textit{irreducible representations} & \xrightarrow{1-1} & P^+ \\
V^\lambda & \longmapsto & \textit{highest weight of } V^\lambda
\end{array}$$

(b) The character of V^λ is

$$\chi^\lambda = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}}{\sum_{w\in W}\varepsilon(w)e^{w\rho}},$$

where $\rho\in P^+$ is defined by $\langle\rho,\alpha_i^\vee\rangle=1$ for $1\le i\le n$ and $\varepsilon(w)=\det(w)$.

(c) The dimension of V^λ is

$$d_\lambda = \frac{\prod_{\alpha>0}\langle\lambda+\rho,\alpha^\vee\rangle}{\prod_{\alpha>0}\langle\rho,\alpha^\vee\rangle}.$$

(d)

$$\chi^\lambda = \sum_{p\in\mathcal{P}_\lambda} e^{p(1)},$$

where $\mathcal{P}_\lambda$ is the set of all paths obtained by acting on p_λ by root operators.

Remark. By part (d)

$$\dim((V^\lambda)_\mu) = \#\text{ paths in } \mathcal{P}_\lambda \text{ which end at } \mu.$$

(For the path model some copying can be done from the Barcelona abstract.)

Remark. Point out that $R(T) = \mathbb{Z}L$, where L is the lattice corresponding to T. Also point out that $R(U) = R(T)^W \cong (\mathbb{Z}L)^W$.