Representation Theory 05.04.2009

A *crystal* is a collection B of paths closed under $\tilde{e}_i, \tilde{f}_i$, for $i = 1, \dots, n$.

Theorem Let B be a crystal. Then

$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C - \rho}} s_{wt(p)},$$

where C is the fundamental chamber with walls $\mathfrak{h}^{\alpha_1^\vee}, \dots, \mathfrak{h}^{\alpha_n^\vee}$ and

$$\begin{aligned} (\mathfrak{h}^*_{\mathbb{Z}})^+ &\longrightarrow (\mathfrak{h}^*_{\mathbb{Z}})^{++} \\ \lambda &\longmapsto \lambda + \rho \end{aligned}$$

is an isomorphism.

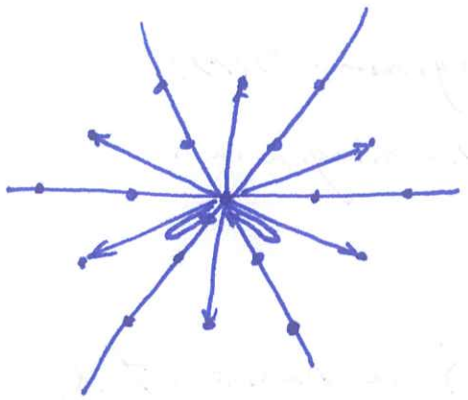
Theorem Let B be a crystal. Let $J \subseteq \{1, \dots, n\}$. By ignoring the action of $\tilde{e}_i, \tilde{f}_i$ for $i \notin J$,

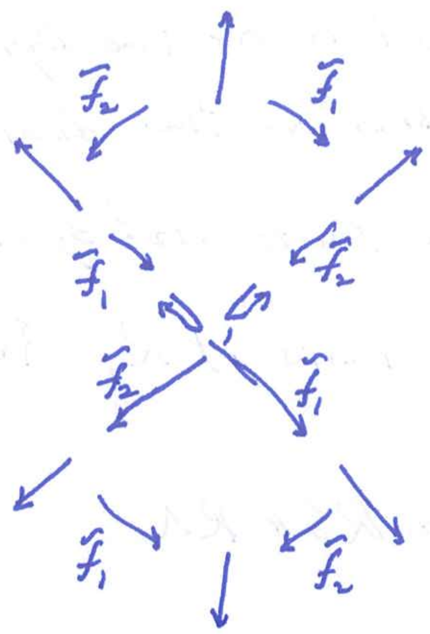
$$B \text{ is a } (W_J, \mathfrak{h}^*_{\mathbb{Z}}) \text{ crystal}$$

where $W_J = \{s_j \mid j \in J\}$. Let $C_J - \rho_J$ be the region on the positive side of $\mathfrak{h}^{\alpha_j^\vee + \delta_j}$ for $j \in J$. Then

$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C_J - \rho_J}} s^J_{wt(p)}.$$

Example Type SL_3 with B given by





Then
$$\mathrm{Res}^{(W,\mathfrak{h}^*_{\mathbb{Z}})}_{(W_J,\mathfrak{h}^*_{\mathbb{Z}})}(B) = B(\Diamond)\sqcup B(\Diamond)\sqcup B(\phi)\sqcup B(\Diamond)$$

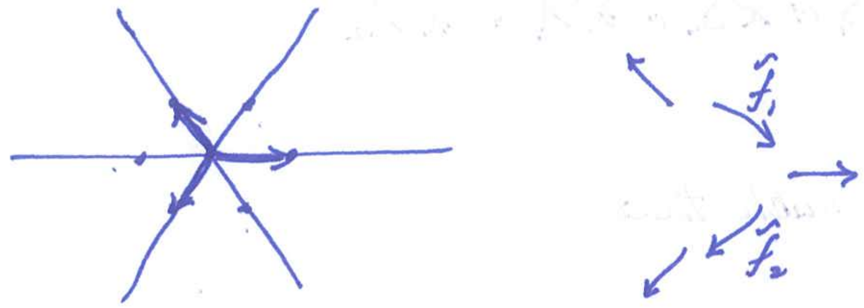
Theorem Let $\lambda, \mu \in (\mathfrak{h}^*_{\mathbb{Z}})^+$. Let $B(\lambda)$ and $B(\mu)$ be irreducible crystals of highest weights λ and μ, respectively. Then

$$B(\lambda)\otimes B(\mu) = \{ p\otimes q \mid p\in B(\lambda),\ q\in B(\mu)\}$$

is a crystal ($p\otimes q$ is the concatenation of p and q). Then

$$s_\lambda s_\mu = \sum_{\substack{q\in B(\mu) \\ p_\lambda^+\otimes q \subseteq C-\rho}} s_{\lambda+wt(q)},$$ where p_λ^+ is the highest weight path in $B(\lambda)$

<u>Example</u> Type SL_3 with B given by



Then

$$s_{\omega_1} s_{\omega_1} = s_{2\omega_1} + s_{\omega_2}$$

since the highest weight paths in $B \otimes B$ are

and

$$s_{\omega_1} s_{\omega_1} s_{\omega_1} = s_{3\omega_1} + 2 s_{\omega_1 + \omega_2} + s_0$$

since the highest weight paths in $B \otimes B \otimes B$ are

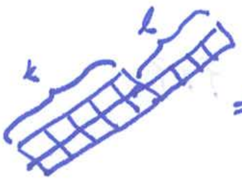
If $\omega_1 = \ydiagram{1}$ and $\omega_2 = \ydiagram{1,1}$ then

$$s_{\omega_1} = s_{\ydiagram{1}}$$

$$s_{\omega_1}^2 = s_{\ydiagram{2}} + s_{\ydiagram{1,1}}$$

$$s_{\omega_1}^3 = s_{\ydiagram{3}} + 2 s_{\ydiagram{2,1}} + s_{\ydiagram{1,1,1}}$$

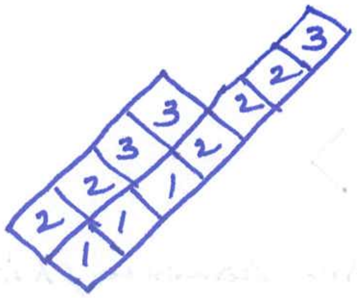
A <u>column strict tableau</u> of shape

$$\lambda = $$  $$= k\omega_2 + l\omega_1$$

is a filling of the boxes of λ from $\{1,2,3\}$ such that

(a) the entries weakly increase from southwest to northeast

(b) the entries strictly increase from southeast to northwest.



Let $p_1 = \nwarrow$, $p_2 = \rightarrow$, $p_3 = \swarrow$ and

$$p_\lambda^+ = \underbrace{p_1 p_1 \cdots p_1}_{k+l} \underbrace{p_2 p_2 \cdots p_2}_{k} \quad \text{if } \lambda = k\omega_2 + l\omega_1$$

There is a bijection from

$$B = \left(\begin{array}{l} \text{irreducible crystal with highest} \\ \text{weight path } p_\lambda^+ \end{array} \right)$$

to

$$B(\lambda) = \left\{ \begin{array}{l} \text{column strict tableaux of} \\ \text{shape } \lambda \text{ filled from } \{1,2,3\} \end{array} \right\}$$

given by reading the tableau from the most northeast box and taking the corresponding word in ρ_1, ρ_2, ρ_3.

<u>Example</u>