Representation Theory Lecture 12 28.10.2008

$$SO_{10} = \{ g \in GL_{10} \mid \det g = 1,\ \ g g^t = 1 \}$$

$$so_{10} = \{ x \in gl_{10} \mid \operatorname{tr} x = 0,\ \ x + x^t = 0 \}$$

since

$$1 = \det(e^{tx}) = \det \begin{pmatrix} e^{th_1} & & \\ & \ddots & \\ & & e^{th_n} \end{pmatrix} = e^{t(h_1 + \cdots + h_n)} = e^{t \cdot \operatorname{tr}(x)}$$

and

$$1 = e^{tx}(e^{tx})^t = e^{tx} e^{tx^t} = e^{t(x+x^t)}$$

Then

$$so_5 = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{pmatrix} \right\}, \quad \dim(so_5) = \frac{5 \cdot 4}{2} = 10$$

Another choice is

$$SO_{10} = \{ g \in GL_{10} \mid \det g = 1,\ \ g J g^t = J \}$$

$$so_{10} = \{ x \in gl_{10} \mid \operatorname{tr} x = 0,\ \ xJ + Jx^t = 0 \}$$

where $J = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}$ or $J = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \end{array}\right)$

Since

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1-3} & a_{1-2} & a_{1-1} \\ a_{21} & a_{22} & a_{23} & a_{2-3} & a_{2-2} & a_{2-1} \\ a_{31} & a_{32} & a_{33} & a_{3-3} & a_{3-2} & a_{3-1} \\ a_{-31} & a_{-32} & a_{-33} & a_{-3-3} & a_{-3-2} & a_{-3-1} \\ a_{-21} & a_{-22} & a_{-23} & a_{-2-3} & a_{-2-2} & a_{-2-1} \\ a_{-11} & a_{-12} & a_{-13} & a_{-1-3} & a_{-1-2} & a_{-1-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{1-1} & a_{1-2} & a_{1-3} & a_{13} & a_{12} & a_{11} \\ a_{2-1} & a_{2-2} & a_{2-3} & a_{23} & a_{22} & a_{21} \\ a_{3-1} & a_{3-2} & a_{3-3} & a_{33} & a_{32} & a_{31} \\ a_{-3-1} & a_{-3-2} & a_{-3-3} & a_{-33} & a_{-32} & a_{-31} \\ a_{-2-1} & a_{-2-2} & a_{-2-3} & a_{-23} & a_{-22} & a_{-21} \\ a_{-1-1} & a_{-1-2} & a_{-1-3} & a_{-13} & a_{-12} & a_{-11} \end{pmatrix}$$

so $\mathfrak{so}_6 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1-3} & a_{1-2} & 0 \\ a_{21} & a_{22} & a_{23} & a_{2-3} & 0 & -a_{1-2} \\ a_{31} & a_{32} & a_{33} & 0 & -a_{2-3} & -a_{1-3} \\ a_{-31} & a_{-32} & 0 & -a_{33} & -a_{23} & -a_{13} \\ a_{-21} & 0 & -a_{-32} & -a_{32} & -a_{22} & -a_{12} \\ 0 & -a_{-21} & -a_{-31} & -a_{31} & -a_{21} & -a_{11} \end{pmatrix}$ with

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & 0 & \\ & & \lambda_3 & & & \\ & & & -\lambda_3 & & \\ & 0 & & & -\lambda_2 & \\ & & & & & -\lambda_1 \end{pmatrix} \right\}$$ and

if $\mathfrak{g} = \mathfrak{so}_6$ then

$$\mathfrak{g} = \mathfrak{h} + a_{12} X_{\varepsilon_1-\varepsilon_2} + a_{13} X_{\varepsilon_1-\varepsilon_3} + a_{23} X_{\varepsilon_2-\varepsilon_3} + a_{21} X_{\varepsilon_2-\varepsilon_1} + a_{31} X_{\varepsilon_3-\varepsilon_1} + a_{32} X_{\varepsilon_3-\varepsilon_2}$$
$$+ a_{1,-2} X_{\varepsilon_1+\varepsilon_2} + a_{1,-3} X_{\varepsilon_1+\varepsilon_3} + a_{2,-3} X_{\varepsilon_2+\varepsilon_3} + a_{-2,1} X_{-(\varepsilon_1+\varepsilon_2)} + a_{-3,1} X_{-(\varepsilon_1+\varepsilon_3)}$$
$$+ a_{-3,2} X_{-(\varepsilon_2+\varepsilon_3)}$$

where, for $i<j$,

$$X_{\varepsilon_i-\varepsilon_j} = E_{ij} - E_{-j,-i}, \qquad X_{\varepsilon_i+\varepsilon_j} = E_{i,-j} - E_{j,-i}$$

$$X_{\varepsilon_j-\varepsilon_i} = E_{ji} - E_{-i,-j}, \qquad X_{-\varepsilon_i-\varepsilon_j} = E_{-j,i} - E_{-i,j}$$

and

$$[h, X_{\varepsilon_i-\varepsilon_j}] = [h, E_{ij} - E_{-j,-i}] = (\lambda_i - \lambda_j) E_{ij} - (-\lambda_j + \lambda_i) E_{-j,-i}$$
$$= (\lambda_i - \lambda_j)(E_{ij} - E_{-j,-i}) = (\lambda_i - \lambda_j) X_{\varepsilon_i-\varepsilon_j}.$$

$$[h, X_{\varepsilon_i+\varepsilon_j}] = [h, E_{i,-j} - E_{j,-i}] = (\lambda_i + \lambda_j) E_{i} - (\lambda_j + \lambda_i) E_{j,-i}$$
$$= (\lambda_i + \lambda_j)(E_{i,-j} - E_{j,-i}) = (\lambda_i + \lambda_j) X_{\varepsilon_i+\varepsilon_j}.$$

if

$$h = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & -\lambda_3 & & \\ & & & & -\lambda_2 & \\ & & & & & -\lambda_1 \end{pmatrix} \quad \text{and}$$

$\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ is given by $\varepsilon_i(h) = \lambda_i$.

The root system

$$R = \{ \pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i \ne j \le r \} \quad \text{and}$$

$$R^+ = \{ \varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le r \}.$$

The Dynkin diagram

The fundamental chamber is

$$(\mathfrak{h}^*_{\mathbb{R}})^+ = \{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle \ge 0 \ \text{for} \ \alpha^\vee \in (R^\vee)^+ \}$$

$$= \{ \lambda_1 \varepsilon_1 + \cdots + \lambda_r \varepsilon_r \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{r-1} \ge |\lambda_r| \}$$

This chamber is on the positive side of the hyperplanes

$$\mathfrak{h}^{\alpha^\vee} = \{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle = 0 \} \quad \text{for} \quad \alpha^\vee \in (R^\vee)^+$$

where

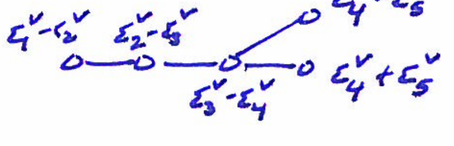
$$(R^\vee)^+ = \{ \varepsilon_i^\vee \pm \varepsilon_j^\vee \mid 1 \le i < j \le r \}. \quad \text{and}$$

$$\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}.$$

The walls of $(\mathfrak{h}^*_{\mathbb{R}})^+$ are

$$\mathfrak{h}^{\varepsilon_1^\vee - \varepsilon_2^\vee}, \ \mathfrak{h}^{\varepsilon_2^\vee - \varepsilon_3^\vee}, \ \ldots, \ \mathfrak{h}^{\varepsilon_{r-1}^\vee - \varepsilon_r^\vee}, \ \mathfrak{h}^{\varepsilon_{r-1}^\vee + \varepsilon_r^\vee}$$

and the Dynkin diagram is



for $\mathfrak{so}_{10}$

The Weylgroup W_0

W_0 is generated by the reflection on the hyperplanes $\mathfrak{h}^{\varepsilon_i^\vee \pm \varepsilon_j^\vee}$. Using the basis $\varepsilon_1, \dots, \varepsilon_r$ for $\mathfrak{h}^*$ the group W_0 is generated by

$$s_{ij} = \left(\begin{array}{c|c} \begin{matrix} 1 & \ddots & 0 & 1 & & \\ & & 1 & 0 & 1 & \ddots & 1 \end{matrix} & \\ \hline & \begin{matrix} 1 & \ddots & 0 & 1 & & \\ & & 1 & 0 & 1 & \ddots & 1 \end{matrix} \end{array}\right) \quad \text{and}$$

$$s_{\varepsilon_i + \varepsilon_j} = \begin{array}{c} i \\ j \\ \\ -j \\ -i \end{array} \left(\begin{array}{c|c} \begin{matrix} 1 & \ddots & 0 & \ddots & 0 & \ddots & 1 \end{matrix} & \begin{matrix} & 1 \\ 1 & \end{matrix} \\ \hline \begin{matrix} & 1 \\ 1 & \end{matrix} & \begin{matrix} 1 & \ddots & 0 & \ddots & 0 & \ddots & 1 \end{matrix} \end{array}\right)$$

$W_0 \cong \left\{ \begin{array}{l} r \times r \text{ matrices with} \\ \text{(a) exactly one nonzero entry in each} \\ \quad \text{row and column} \\ \text{(b) nonzero entries are } \pm 1. \\ \text{(c) } \prod\limits_{\substack{\text{nonzero} \\ \text{entries}}} a_{ij} = 1 \end{array} \right\}$

The character of the adjoint representation $\mathfrak{g}$

Let M be a $\mathfrak{g}$-module.

The character of M is

$$\mathrm{char}(M) = \sum_{\mu \in \mathfrak{h}^*} (\dim M_\mu) e^\mu, \quad \text{where}$$

$$M_\mu = \{ m \in M \mid hm = \mu(h) m \text{ for each } h \in \mathfrak{h} \}.$$

is the μ-weight space of M.

The weights of the adjoint representation for $\mathfrak{g} = \mathfrak{so}_{10}$ are

$$\begin{array}{cccccccc}
\varepsilon_1-\varepsilon_2 & \varepsilon_1-\varepsilon_3 & \varepsilon_1-\varepsilon_4 & \varepsilon_1-\varepsilon_5 & \varepsilon_1+\varepsilon_5 & \varepsilon_1+\varepsilon_4 & \varepsilon_1+\varepsilon_3 & \varepsilon_1+\varepsilon_2 \\
& \varepsilon_2-\varepsilon_3 & \varepsilon_2-\varepsilon_4 & \varepsilon_2-\varepsilon_5 & \varepsilon_2+\varepsilon_5 & \varepsilon_2+\varepsilon_4 & \varepsilon_2+\varepsilon_3 & \\
& & \varepsilon_3-\varepsilon_4 & \varepsilon_3-\varepsilon_5 & \varepsilon_3+\varepsilon_5 & \varepsilon_3+\varepsilon_4 & & \\
& & & \varepsilon_4-\varepsilon_5 & \varepsilon_4+\varepsilon_5 & & &
\end{array}$$

their negatives and the weight 0:

$$\mathfrak{g}_0 = \mathfrak{h} \text{ and } \dim(\mathfrak{g}_0) = \dim(\mathfrak{h}) = 5.$$

The character of $\mathfrak{g}$ is

$$\begin{aligned}
s_{\varepsilon_1+\varepsilon_2} = {} & x_1x_2^{-1} + x_1x_3^{-1} + x_1x_4^{-1} + x_1x_5^{-1} + x_1x_5 + x_1x_4 + x_1x_3 + x_1x_2 \\
& + x_2x_3^{-1} + x_2x_4^{-1} + x_2x_5^{-1} + x_2x_5 + x_2x_4 + x_2x_3 \\
& + x_3x_4^{-1} + x_3x_5^{-1} + x_3x_5 + x_3x_4 \\
& + x_4x_5^{-1} + x_4x_5 \\
& + 5 \\
& + x_1^{-1}x_2 + x_1^{-1}x_3 + x_1^{-1}x_4 + x_1^{-1}x_5 + x_1^{-1}x_5^{-1} + x_1^{-1}x_4^{-1} + x_1^{-1}x_3^{-1} + x_1^{-1}x_2^{-1}
\end{aligned}$$

$$+ x_2^{-1}x_3 + x_2^{-1}x_4 + x_2^{-1}x_5 + x_2^{-1}x_5^{-1} + x_2^{-1}x_4^{-1} + x_2^{-1}x_3^{-1}$$
$$+ x_3^{-1}x_4 + x_3^{-1}x_5 + x_3^{-1}x_5^{-1} + x_3^{-1}x_4^{-1}$$
$$+ x_4^{-1}x_5 + x_4^{-1}x_5^{-1}$$

where
$$x_i = e^{\varepsilon_i}, \quad \text{for} \quad i = 1, 2, \ldots, 5.$$

In this example
$$\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \frac{1}{2}(8\varepsilon_1 + 6\varepsilon_2 + 4\varepsilon_3 + 2\varepsilon_4) = 4\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4.$$

The Weyl denominator formula says
$$a_\rho = \sum_{w \in W_0} \det(w)\, w(x_1^4 x_2^3 x_3^2 x_4) = x_1^4 x_2^3 x_3^2 x_4 + x_1^{-4} x_2^3 x_3^2 x_4 + \cdots + x_4^{-4} x_3^{-3} x_2^{-2} x_1^{-1}$$
$$= e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = x_1^4 x_2^3 x_3^2 x_4 \prod_{i<j} (1 - x_i^{-1}x_j)(1 - x_i^{-1}x_j^{-1})$$

and the Weyl character formula says
$$s_{\varepsilon_1+\varepsilon_2} = \frac{a_{\varepsilon_1+\varepsilon_2+\rho}}{a_\rho} = \frac{\sum_{w \in W_0} \det(w)\, w(x_1^5 x_2^4 x_3^2 x_4)}{x_1^4 x_2^3 x_3^2 x_4 \prod_{1 \le i < j \le 5} (1 - x_i^{-1}x_j)(1 - x_i^{-1}x_j^{-1})}$$

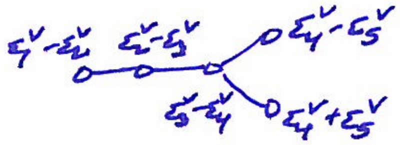
The crystal $B(\varepsilon_1+\varepsilon_2)$

$\mathfrak{h}_{\mathbb{R}}^*$ has basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$
and crystals are sets of paths in $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^5$
which are closed under the action of the
root operators

$$\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{f}_5$$

corresponding to the walls of $(\mathfrak{h}_{\mathbb{R}}^*)^+$:

$$\mathfrak{h}^{\varepsilon_1^\vee-\varepsilon_2^\vee}, \mathfrak{h}^{\varepsilon_2^\vee-\varepsilon_3^\vee}, \mathfrak{h}^{\varepsilon_3^\vee-\varepsilon_4^\vee}, \mathfrak{h}^{\varepsilon_4^\vee-\varepsilon_5^\vee}, \mathfrak{h}^{\varepsilon_4^\vee+\varepsilon_5^\vee}$$



The weights of $\mathfrak{g}$ are $\pm(\varepsilon_i \pm \varepsilon_j)$, $1 \le i < j \le 5$
and these are some of the vertices of the 5 dimensional cube

The highest weight path in $B(\varepsilon_1+\varepsilon_2)$ can be
taken to be the straight line path from
0 to $\varepsilon_1+\varepsilon_2$.

Most of the time the root operators (in $B(\varepsilon_1+\varepsilon_2)$) are
taking a straight line path to a straight line
path. The only exceptions are

$$p_{\varepsilon_4+\varepsilon_5} \xrightarrow{\tilde{f}_5} \left(\tfrac{1}{2} p_{-\varepsilon_4-\varepsilon_5}\right) \otimes \left(\tfrac{1}{2} p_{\varepsilon_4+\varepsilon_5}\right) \xrightarrow{\tilde{f}_5} p_{-\varepsilon_4-\varepsilon_5}$$

$$p_{\varepsilon_4-\varepsilon_5} \xrightarrow{\tilde{f}_4} \left(\tfrac{1}{2} p_{-\varepsilon_4+\varepsilon_5}\right) \otimes \left(\tfrac{1}{2} p_{\varepsilon_4-\varepsilon_5}\right) \xrightarrow{\tilde{f}_4} p_{-\varepsilon_4+\varepsilon_5}$$

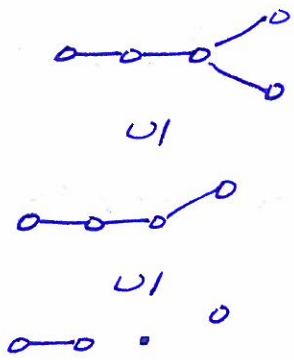
$$p_{\varepsilon_3-\varepsilon_4} \underset{\tilde e_3}{\overset{\tilde f_3}{\rightleftarrows}} \left(\tfrac12 p_{-\varepsilon_3+\varepsilon_4}\right)\otimes\left(\tfrac12 p_{\varepsilon_3-\varepsilon_4}\right) \underset{\tilde e_3}{\overset{\tilde f_3}{\rightleftarrows}} p_{-\varepsilon_3+\varepsilon_4}$$

$$p_{\varepsilon_2-\varepsilon_3} \underset{\tilde e_2}{\overset{\tilde f_2}{\rightleftarrows}} \left(\tfrac12 p_{-\varepsilon_2+\varepsilon_3}\right)\otimes \tfrac12\left(p_{\varepsilon_2-\varepsilon_3}\right) \underset{\tilde e_2}{\overset{\tilde f_2}{\rightleftarrows}} p_{-\varepsilon_2+\varepsilon_3}$$

$$p_{\varepsilon_1-\varepsilon_2} \underset{\tilde e_1}{\overset{\tilde f_1}{\rightleftarrows}} \left(\tfrac12 p_{-\varepsilon_1+\varepsilon_2}\right)\otimes\left(\tfrac12 p_{\varepsilon_1-\varepsilon_2}\right) \underset{\tilde e_1}{\overset{\tilde f_1}{\rightleftarrows}} p_{-\varepsilon_1+\varepsilon_2}$$

For the "standard model" in particle physics it is important to understand how this representation decomposes under the action of the subalgebras

$$SO_{10}$$
$$\cup\!\!\mid$$
$$SU_5$$
$$\cup\!\!\mid$$
$$SU_3\times U_1\times SU_2$$



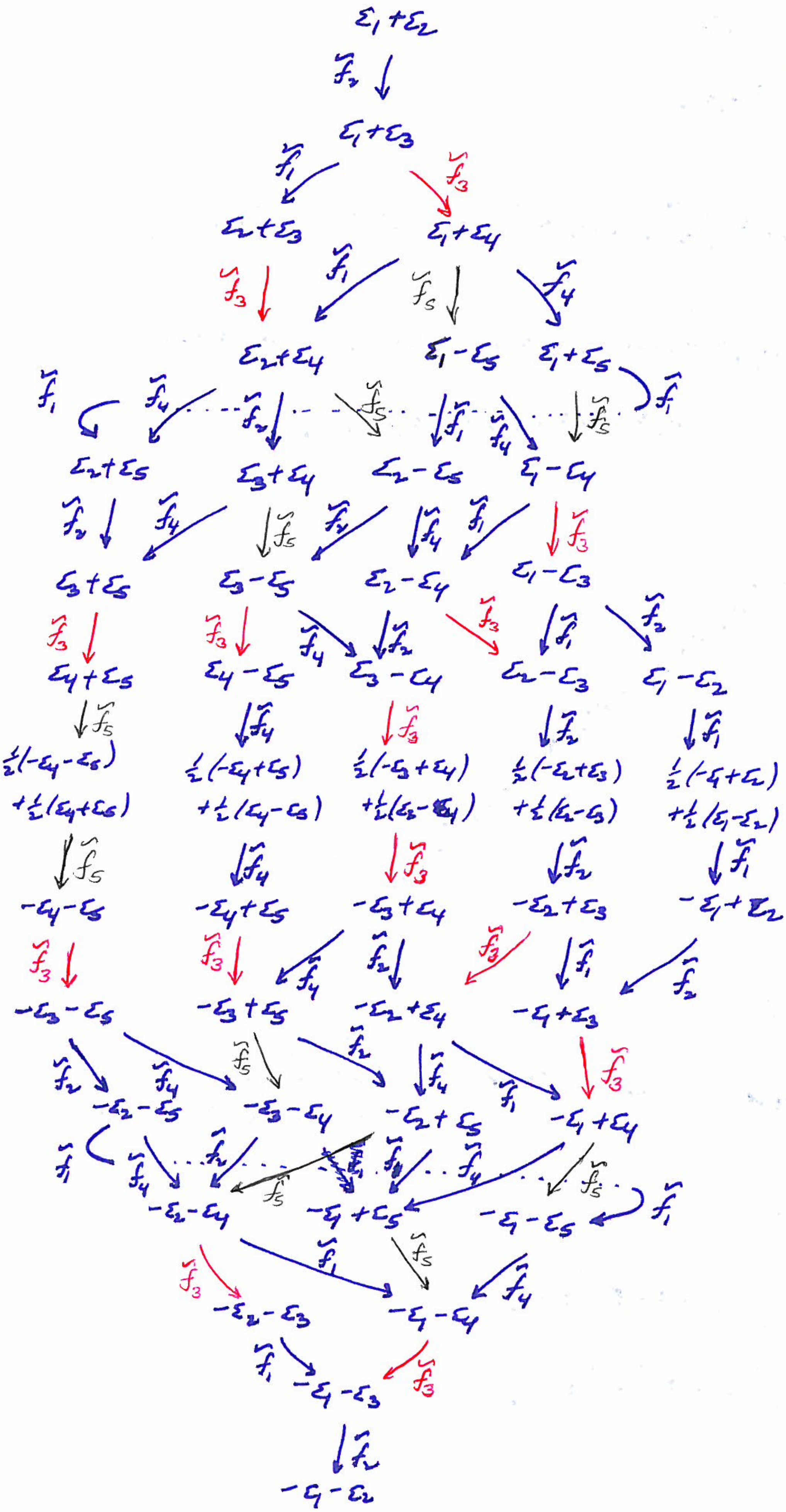
These restrictions are obtained by ignoring the operators

$\tilde f_6$, for $SO_{10}\supseteq SU_5$,

and $\tilde f_3$, for $SU_5\supseteq SU_3\times U_1\times SU_2$.

The crystal graph $B(\varepsilon_1+\varepsilon_2)$ is (all paths are straight line paths except the 5 exceptional ones listed above):

The crystal graph of $B(\varepsilon_1+\varepsilon_2)$



$$
\begin{tikzcd}
\varepsilon_4-\varepsilon_5 \arrow[d, "\check{f}_4"] \\
\frac{1}{2}(-\varepsilon_4-\varepsilon_5)+\frac{1}{2}(\varepsilon_4-\varepsilon_5) \arrow[d, "\check{f}_4"] \\
-\varepsilon_4+\varepsilon_5
\end{tikzcd}
\qquad
\begin{matrix}
\frac{1}{2}(-\varepsilon_3+\varepsilon_4) \\
+\frac{1}{2}(\varepsilon_3-\varepsilon_4)
\end{matrix}
\qquad
\begin{tikzcd}
\varepsilon_1-\varepsilon_3 \arrow[d, "\check{f}_1"] \arrow[dr, "\check{f}_2"] & \\
\varepsilon_2-\varepsilon_3 \arrow[d, "\check{f}_2"] & \varepsilon_1-\varepsilon_2 \arrow[d, "\check{f}_1"] \\
\frac{1}{2}(-\varepsilon_2+\varepsilon_3)+\frac{1}{2}(\varepsilon_2-\varepsilon_3) \arrow[d, "\check{f}_2"] & \frac{1}{2}(-\varepsilon_1+\varepsilon_2)+\frac{1}{2}(\varepsilon_1-\varepsilon_2) \arrow[d, "\check{f}_1"] \\
-\varepsilon_2+\varepsilon_3 \arrow[d, "\check{f}_1"] & -\varepsilon_1+\varepsilon_2 \arrow[dl, "\check{f}_2"] \\
-\varepsilon_1+\varepsilon_3 &
\end{tikzcd}
$$

3 bosons
W^+, Z, W^-

1 photon

8 gluons of QCD