REPRESENTATION THEORY

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Abstract. Notes from Arun Ram's 2008 course at the University of Melbourne.

8. Week 8

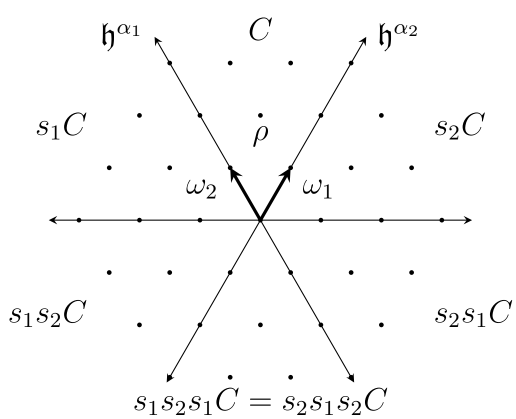
Setup: We start with a lattice $\mathfrak{h}^*_{\mathbb{Z}}$ (a $\mathbb{Z}$-vector space), and $W_0 \subset GL(\mathfrak{h}^*_{\mathbb{Z}})$, a finite subgroup generated by reflections: the reflections in W_0 are s_α, $\alpha \in R^+$ with

$$s_\alpha \mu = mu - \langle \mu, \alpha^\vee \rangle \, \alpha$$

for $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$.

Fix a fundamental region C for the action of W_0 on $\mathfrak{h}^*_{\mathbb{R}}$. Let $\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ be the walls of C and the reflections in these are $s_1, \ldots, s_n$, the simple reflections. Recall $P^+ = \mathfrak{h}^*_{\mathbb{Z}} \cap \bar{C}$ and $P^{++} = \mathfrak{h}^*_{\mathbb{Z}} \cap C$.

You should have a picture in your head of this, for example SL_3, where $\mathfrak{h}^*_{\mathbb{Z}} = \text{span}\,\{\omega_1, \omega_2\}$:



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Recall that

$$W_0 \longleftrightarrow \{\text{fundamental regions}\}$$

and

$$\begin{aligned} P^+ &\xrightarrow{\sim} P^{++} \\ \lambda &\longmapsto \lambda + \rho \end{aligned}$$

is an isomorphism of semigroups.

We take $\mathbb{C}[X] = \text{span}\{X^\mu | \mu \in \mathfrak{h}^*_\mathbb{Z}\}$ with $X^\mu X^\nu = X^{\mu+\nu}$, with W_0 acting on $\mathbb{C}[X]$ by $wX^\mu = X^{w\mu}$, and recall

$$\begin{aligned} \mathbb{C}[X]^{W_0} &= \{f \in \mathbb{C}[X] | wf = f \text{ for all } w \in W_0\} \\ \mathbb{C}[X]^{\det} &= \{f \in \mathbb{C}[X] | wf = \det w \cdot f \text{ for all } w \in W_0\} \end{aligned}$$

The second of these has basis

$$a_{\lambda+\rho} = \sum_{w \in W_0} \det w^{-1} \cdot X^{w(\lambda+\rho)},$$

for $\lambda \in P^+$, ρ the cone point of P^{++}.

Theorem 8.1 (The boson-fermion correspondence)**.** *As $\mathbb{C}[X]^{W_0}$-modules,*

$$\begin{aligned} \Phi : \mathbb{C}[X]^{W_0} &\xrightarrow{\sim} \mathbb{C}[X]^{\det} \\ f &\longmapsto a_\rho f \end{aligned}$$

is an isomorphism.

The element a_ρ is the *Weyl denominator*, or the *Vandermonde*, defined as above; for example, in SL_3,

$$a_\rho = X^\rho - X^{s_1\rho} - X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} - X^{s_1s_2s_1\rho}.$$

Proof. (a) Φ is a $\mathbb{C}[X]^{W_0}$-module homomorphism: If $g \in \mathbb{C}[X]^{W_0}$ then

$$\Phi(gf) = a_\rho gf = g a_\rho f = g\Phi(f).$$

(b) Φ is well-defined, ie $\Phi(f) \in \mathbb{C}[X]^{\det}$: If $w \in W_0$ then

$$w\Phi(f) = w(a_\rho f) = (wa_\rho)(wf) = \det w \cdot a_\rho f = \det w \cdot \Phi(f),$$

since $w(X^\mu X^\nu) = w(X^{\mu+\nu}) = X^{w(\mu+\nu)} = X^{w\mu+w\nu} = (wX^\mu)(wX^\nu)$

(c) Φ is invertible: We have to show that if $g \in \mathbb{C}[X]^{\det}$ then g is divisible by a_ρ, and aslo that $\frac{g}{a_\rho}$ is symmetric. The second of these is easy to check. To see that $a_\rho | g$, take $g \in \mathbb{C}[X]^{\det}$ and let s_α be a reflection in W_0 (so $s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha$, $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}$).

Since $s_\alpha g = \det s_\alpha \cdot g = -g$, we know

$$g = \frac{1}{2}(g - s_\alpha g) = \frac{1}{2}(1 - s_\alpha) g$$

and we can expand g in the X^μ basis:

$$\begin{aligned} &= \frac{1}{2}(1 - s_\alpha) \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu X^\mu = \frac{1}{2} \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu (X^\mu - X^{s_\alpha \mu}) \\ &= \frac{1}{2} \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu (X^\mu - X^{\mu - \langle \mu, \alpha^\vee \rangle \alpha}) = \frac{1}{2} \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu X^\mu (1 - X^{-\langle \mu, \alpha^\vee \rangle \alpha}) \end{aligned}$$

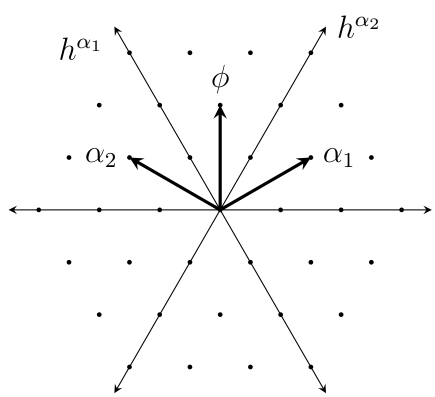
and, as $(1 - X^{k\alpha})$ is divisible by $(1 - X^{-\alpha})$,[1] we get that $1 - X^{-\alpha}$ divides $g = \frac{1}{2} \sum_{\mu \in \mathfrak{h}_{\mathbb{Z}}^*} g_\mu X^\mu (1 - X^{-\langle \mu, \alpha^\vee \rangle \alpha})$.

The $1 - X^{-\alpha}$ are relatively prime in $\mathbb{C}[X]$ and so g is divisible by $\prod_{\alpha \in R^+}(1 - X^{-\alpha})$. In particular, $a_\rho \in \mathbb{C}[X]^{W_0}$ and is divisible by $\prod_{\alpha \in R^+}(1 - X^{-\alpha})$.

Claim: $a_\rho = (\prod_{\alpha \in R^+} X^{\alpha/2})(\prod_{\alpha \in R^+}(1 - X^{-\alpha}))$

This is because $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$,

(For example:)



[1]for example, $(1 - X^{-4\alpha}) = (1 - X^{-\alpha})(1 + X^{-\alpha} + X^{-2\alpha} + X^{-3\alpha})$ and $(1 - X^{4\alpha}) = -X^{4\alpha}(1 - X^{-4\alpha}) = -X^{4\alpha}(1 - X^{-\alpha})(1 + X^{-\alpha} + X^{-2\alpha} + X^{-3\alpha})$

and also for the following geometric reasons:

(1) s_i permutes $R^+ - \{\alpha_i\}$. (C is on the positive side of all hyperplanes; $s_1 C$ is on the positive side of $\mathfrak{h}^{\alpha^\vee}$ for all $\alpha \in R^+$ except α_1. Note that this means that this fact is very perculiar to real reflection groups.)

(2) w_0, the longest element of W_0, sends R^+ to $R^- = -R^+$ (This is again a geometric fact; $w_0 C$ is the unique chamber on the negative side of all hyperplanes)

Note that

$$\begin{aligned} RHS &= \prod_{\alpha \in R^+} X^{\alpha/2} + \cdots \text{stuff} + \prod_{\alpha \in R^+} X^{-\alpha/2} \\ &= X^\rho + \cdots \text{stuff} + X^{-\rho} \\ &= a_\rho \end{aligned}$$

and so $a_\rho = X^\rho \prod_{\alpha \in R^+}(1 - X^{-\alpha})$; this is *Weyl's denominator formula.* Thus our claim is proved. □

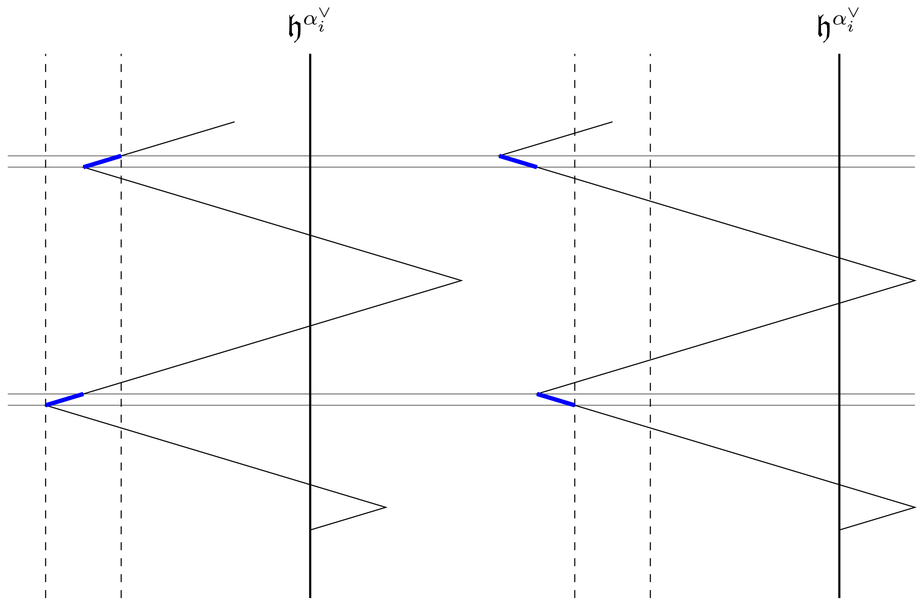
Remark. For type GL_n, Weyl's denominator formula is

$$a_\rho = \det \begin{pmatrix} X_1^{n-1} & X_1^{n-2} & \cdots & X_1 & 1 \\ X_2^{n-1} & X_2^{n-2} & \cdots & X_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ X_n^{n-1} & X_n^{n-2} & \cdots & X_n & 1 \end{pmatrix} = \prod_{i<j}(X_i - X_j)$$

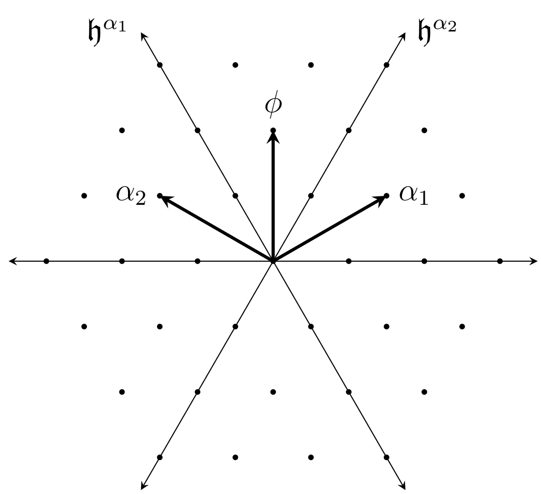
8.1. **Crystals and symmetric functions.** $\mathbb{C}[X]^{W_0}$ are really characters of crystals.

Definition. A *path* is a function $p : [0,1] \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$ (piecewise linear, say) such that $p(0) = 0$ and $p(1) \in \mathfrak{h}^*_{\mathbb{Z}}$.

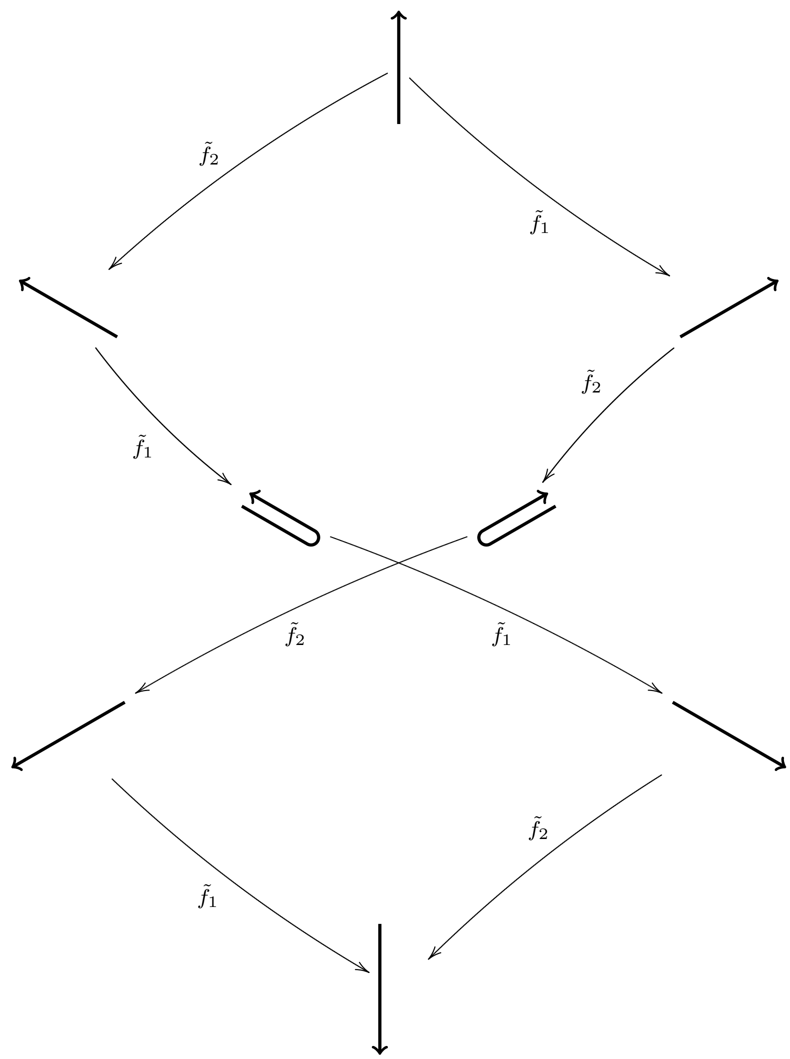
Definition. A *crystal* is a set of paths B which is closed under the action of the root operators $\tilde{e}$ and $\tilde{f}$:



The process illustrated above is to draw a dotted line (parallel to $\mathfrak{h}^{\alpha_i^\vee}$) along the rightmost point of your path, draw another parallel line which is d_i to the left of it (where d_i is the distance between parallel lines of lattice points), then pour water into this region and see which portions of the path get wet (the blue segments above). To create a new path, reproduce the old path but reflect the wet (blue) segments, translating the rest of the path as necessary.



Starting with the path ϕ, we can build a crystal:



The *character* of a crystal B is

$$\mathrm{char}(B) = \sum_{p \in B} X^{\mathrm{wt}(p)},$$

where $\mathrm{wt}(p)$ is the endpoint of p.

For example, the character of the above crystal is $X^{\rho} + X^{s_1\rho} + X^{s_2\rho} + X^{s_1 s_2\rho} + X^{s_2 s_1\rho} + X^{s_1 s_2 s_1\rho} + 2X^0$.

So we're seeing that symmetric functions are shadows of crystals.

We want to see if $\mathrm{char}(B) \in \mathbb{C}[X]^{W_0}$ in more than just this example.

Definition. Let $p \in B$. The *i-string* of p is

$$\tilde{f}_i^r p - \cdots - \tilde{f}_i^2 p - \tilde{f}_i p - p - \tilde{e}_i p - \tilde{e}_i^2 p - \cdots - \tilde{e}_i^s p$$

(read "edge" not "minus" for $-$) where $\tilde{f}_i^{r+1} p = 0$ and $\tilde{e}_i^{s+1} p = 0$.

$\tilde{e}_i^s p$ is the head of the i-string of p; if $h = \tilde{e}_i^s p$ then we rewrite the string as

$$\tilde{f}_i^{\langle \mu, \alpha_i^\vee \rangle} h - \cdots - \tilde{f}_i^2 h - \tilde{f}_i h - h.$$

If the weight of h is μ then the elements of this string have weights $s_i \mu = \mu - \langle \mu, \alpha_i^\vee \rangle, \ldots, \mu - 2\alpha_1, \mu - \alpha_i, \mu$.

Define an actions of W_0 on B by $s_i p$ is the opposite of p in its i-string. So s_i flips the whole crystal.

Then $\mathrm{wt}(s_i p) = s_i \mathrm{wt}(p)$; So $s_i \mathrm{char}(B) = \mathrm{char}(s_i B) = \mathrm{char}(B)$ and $\mathrm{char}(B) \in \mathbb{C}[X]^{W_0}$.

An irreducible crystal is a crystal B such that the crystal graph is connected.

What are the characters of irreducibles?

Definition. The *Weyl characters*, or *Schur functions*, are $s_\lambda = a_{\lambda+\rho}/a_\rho$, $\lambda \in P^+$.

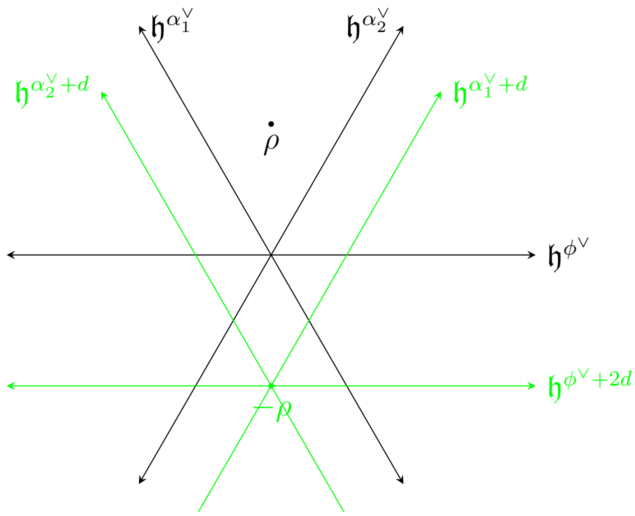
So the s_λ are the images of $a_{\lambda+\rho}$ under the "divide by a_ρ" isomorphism,

$$\begin{aligned} \mathbb{C}[X]^{\det} &\xrightarrow{\sim} \mathbb{C}[X]^{W_0} \\ a_{\lambda+\rho} &\longmapsto s_\lambda. \end{aligned}$$

Definition. The dot action of W_0 on $\mathfrak{h}_\mathbb{Z}^*$ is

$$w \circ \mu := w(\mu + \rho) - \rho \text{ for } \mu \in \mathfrak{h}_\mathbb{Z}^*, \, w \in W_0.$$

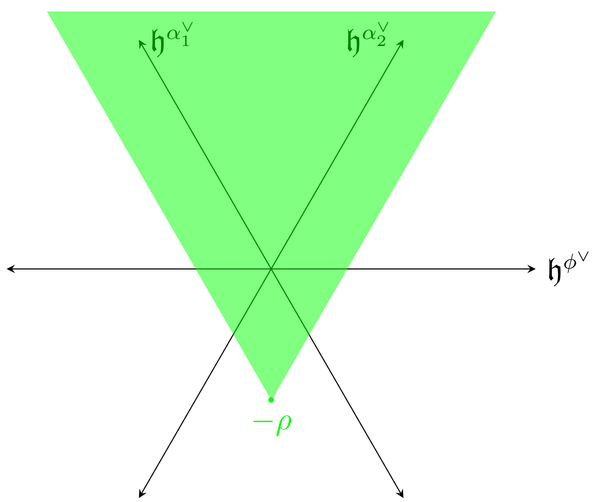
We can see $w \circ (-\rho) = w(-\rho + \rho) - \rho) = 0 - \rho = -\rho$, so the planes of reflection pass through $-\rho$, for example:



Recall $s_\mu = \frac{a_{\mu+\rho}}{a_\rho}$ for all $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$. Then

$$s_{w\circ\mu} = s_{w(\mu+\rho)-\rho} = \frac{a_{w(\mu+\rho)} - \rho + \rho}{a_\rho} = \frac{a_{w(\mu+\rho)}}{a_\rho} = \det w \frac{a_{\mu+\rho}}{a_\rho} = \det w \cdot s_\mu.$$

Definition. A highest weight path is $p \subset C - \rho$.



p is highest weight if and only if $\tilde{e}_i p = 0$ for all i. Each irreducible crystal has a unique highest weight path.

Theorem 8.2. *Let B be a crystal. Then*

$$char(B) = \sum_{p \in B, p \subset C - \rho} s_{wt(p)}.$$

We'll prove this next week. Meanwhile please enjoy the following corollaries:

Corollary 8.3 (Weyl character formula)**.** *Let p_λ^+ be a highest weight path with $wt(p_\lambda^+) = \lambda$. Let $B(\lambda)$ be the crystal generated by p_λ^+. Then $char(B(\lambda)) = s_\lambda$.*

Corollary 8.4 (Littlewood-Richardson rule)**.** *(1994 in this generality; L-R 1935)*

$$char(B(\lambda) \otimes B(\mu)) = \sum_{p_\lambda^+ \otimes q \subset C - \rho} s_{wt(p_\lambda^+ \otimes q)} = \sum_{q \in B(\mu), p_\lambda^+ \otimes q \subset C - \rho} s_{\lambda + wt(q)}$$

where $B(\lambda) \otimes B(\mu) = \{p \otimes q | p \in B(\lambda), q \in B(\mu)\}$