Set up

$\mathfrak{h}^*_{\mathbb{Z}}$ is a $\mathbb{Z}$-vector space.

$W_0 \subseteq GL(\mathfrak{h}^*_{\mathbb{Z}})$ a finite group gen. by reflections

s_α, $\alpha \in R^+$, are the reflections in W_0,

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha$$

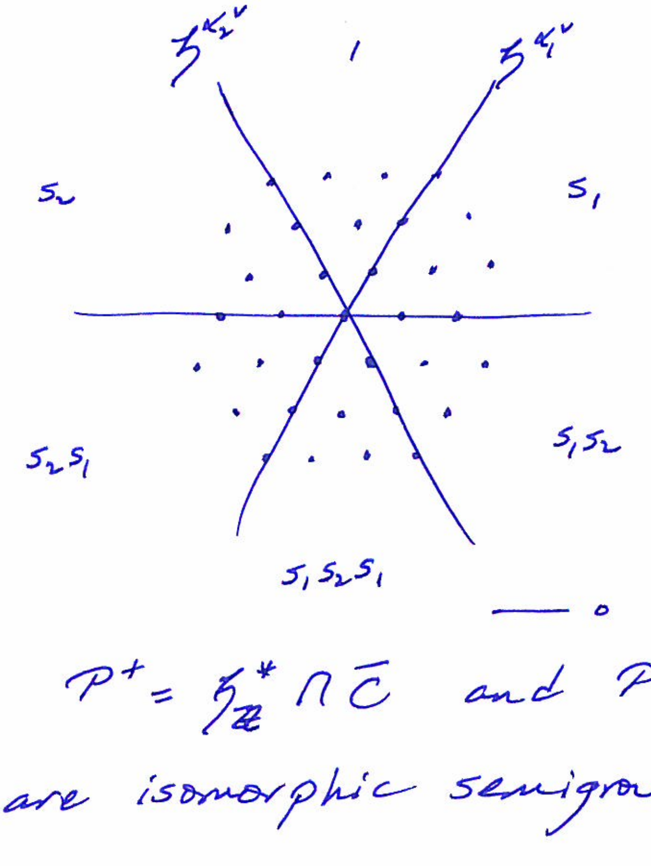
C is a fundamental chamber for W_0 acting on $\mathfrak{h}^*_{\mathbb{R}}$

$\mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ are the walls of C and their reflections

$s_1, \ldots, s_n$ are the simple reflections.

$$W_0 \xleftrightarrow{1-1} \{\text{chambers on } \mathfrak{h}^*_{\mathbb{R}}\}$$

Example Type SL_3 $\quad \mathfrak{h}^*_{\mathbb{Z}} = \text{span}\{\omega_1, \omega_2\}$



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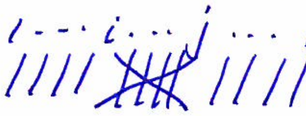
$P^+ = \mathfrak{h}^*_{\mathbb{Z}} \cap \overline{C}$ and $P^{++} = \mathfrak{h}^*_{\mathbb{Z}} \cap C$

are isomorphic semigroups $\quad P^+ \xrightarrow{\sim} P^{++}$

$$\lambda \longmapsto \lambda + \rho$$

<u>Example</u> <u>Type GL_n</u> $\mathfrak{h}^*_{\mathbb{Z}} = \text{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ and

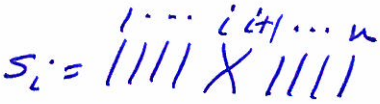
$W_0 = S_n$, which has reflections

$s_{ij} =$  with $\mathfrak{h}^{ij} = \{\mu = (\mu_1, \ldots, \mu_n) \mid \mu_i = \mu_j\}$.

Then

$$C = \{\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n \mid \mu_1 > \mu_2 > \cdots > \mu_n\}$$

has walls

$\mathfrak{h}^{\alpha_i^\vee} = \{\mu = (\mu_1, \ldots, \mu_n) \mid \mu_i = \mu_{i+1}\}$ and $s_i =$ 

Then

$$P^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$$

$$P^{++} = \{\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 > \cdots > \mu_n\}$$

and

$$\begin{aligned} P^+ &\longrightarrow P^{++} \\ \lambda &\longmapsto \lambda + \rho \quad \text{where } \rho = (n-1, n-2, \ldots, 1, 0). \end{aligned}$$

$$\mathbb{C}[X] = \text{span}\{X^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\} \text{ with } X^\mu X^\nu = X^{\mu+\nu}$$

$$\mathbb{C}[X]^{W_0} = \{f \in \mathbb{C}[X] \mid wf = f, \text{ for all } w \in W_0\}$$

$$\mathbb{C}[X]^{\det} = \{f \in \mathbb{C}[X] \mid wf = \det(w)f, \text{ for all } w \in W_0\}.$$

Then $\mathbb{C}[X]^{W_0}$ has basis

$$m_\lambda = \sum_{\gamma \in W_0\lambda} X^\gamma, \qquad \lambda \in P^+$$

$\mathbb{C}[X]^{\det}$ has basis

$$a_{\lambda+\rho} = \sum_{w \in W_0} \det(w) X^{w(\lambda+\rho)}.$$

<u>Theorem</u> As $\mathbb{C}[X]^{W_0}$-modules

$$\begin{aligned} \Phi : \mathbb{C}[X]^{W_0} &\xrightarrow{\sim} \mathbb{C}[X]^{\det} \\ f &\longmapsto a_\rho f. \end{aligned}$$

<u>Proof</u> (a) Φ is a $\mathbb{C}[X]^{W_0}$-homomorphism.

If $g \in \mathbb{C}[X]^{W_0}$ then

$$\Phi(gf) = a_\rho g f = g a_\rho f = g \Phi(f).$$

(b) Φ is well defined.

If $w \in W_0$ then

$$\begin{aligned} w\Phi(f) = w a_\rho f = (w a_\rho)(wf) &= \det(w) a_\rho f \\ &= \det(w)\Phi(f). \end{aligned}$$

(c) Φ is invertible.

Let $g \in \mathbb{C}[X]^{\det}$, $\quad g = \sum_{\mu \in \mathfrak{h}^*_{\mathbb{Z}}} g_\mu X^\mu.$

~~So~~ Let s_α be a reflection in W_0,

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha, \quad \text{with } \langle \mu, \alpha^\vee \rangle \in \mathbb{Z}.$$

Then
$s_\alpha g = \det(s_\alpha) g = -g$, so that $g = \frac{1}{2}(g - s_\alpha g)$

So

$$g = \frac{1}{2}(1-s_\alpha)g = \frac{1}{2}\sum_{\mu \in \mathfrak{h}^*_{\mathbb{Z}}} g_\mu (X^\mu - X^{s_\alpha \mu})$$

$$= \frac{1}{2}\sum_{\mu \in \mathfrak{h}^*_{\mathbb{Z}}} g_\mu X^\mu (1 - X^{-\langle \mu, \alpha^\vee\rangle \alpha})$$

Note, for example,

$$1 - X^{-5\alpha} = (1-X^{-\alpha})(1+X^\alpha + X^{2\alpha} + X^{3\alpha} + X^{4\alpha})$$

$$1 - X^{5\alpha} = X^{5\alpha}(1-X^{-\alpha})(1+X^\alpha + X^{2\alpha} + X^{3\alpha} + X^{4\alpha})(-1)$$

In any case, g is divisible by $1-X^{-\alpha}$.
Since $1-X^{-\alpha}$, $\alpha \in R^+$, are relatively prime

g is divisible by $\prod_{\alpha \in R^+}(1-X^{-\alpha})$.

<u>Note:</u>

$$a_\rho = \left(\prod_{\alpha \in R^+} X^{\alpha/2}\right)\prod_{\alpha \in R^+}(1-X^{-\alpha})$$

$$= X^\rho \prod_{\alpha \in R^+}(1-X^{-\alpha})$$

since

$$a_\rho = X^\rho + \cdots + X^{w_0\rho} \quad \text{and} \quad \rho = \frac{1}{2}\sum_{\alpha \in R^+}\alpha$$

because

(a) s_i permutes $R^+ - \{\alpha_i\}$ and $s_i\alpha_i = -\alpha_i$

(b) w_0 sends $R^+ = \{\alpha\}$ to $R^- = \{-\alpha \mid \alpha \in R^+\}$.

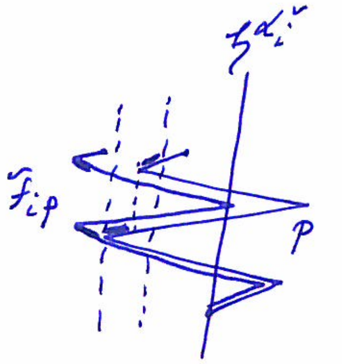
The <u>Weyl character</u>, or <u>Schur function</u> is

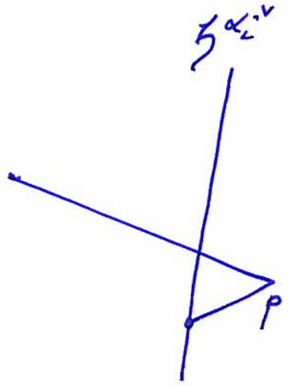
$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{so that} \quad \mathbb{C}[X]^{W_0} \longrightarrow \mathbb{C}[X]^{\det}$$
$$s_\lambda \longmapsto a_{\lambda+\rho}$$

Crystals

A <u>path</u> is $p : [0,1] \longrightarrow \mathfrak{h}_{\mathbb{R}}^*$ (piecewise linear) with $p(0) = 0$ and $p(1) \in \mathfrak{h}_{\mathbb{Z}}^*$.

A <u>crystal</u> is a set of paths B, closed under the action of the root operators $\tilde{e}_i, \tilde{f}_i$.





$\tilde{f}_i p = 0$

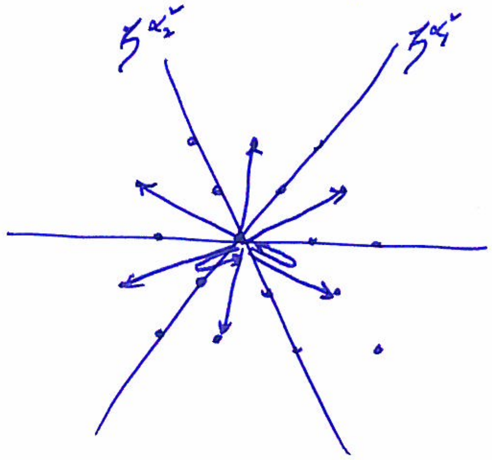
and

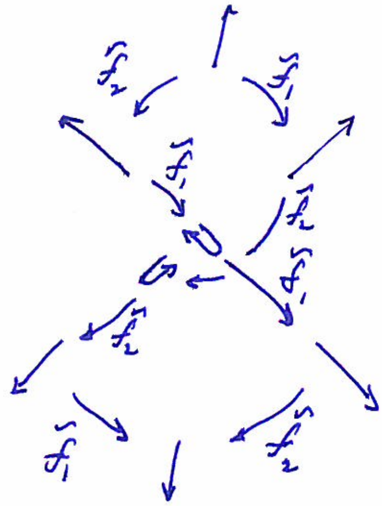
$$\tilde{e}_i \tilde{f}_i p = p, \text{ if } \tilde{f}_i p \neq 0 \quad \text{and} \quad \tilde{f}_i \tilde{e}_i p = p, \text{ if } \tilde{e}_i p \neq 0.$$

The <u>character</u> of B is

$$\mathrm{char}(B) = \sum_{p \in B} X^{wt(p)}.$$

Favourite example





Let B be a crystal ~~and~~ let $p \in B$.

The <u>i-string</u> of p is

$$\tilde f_i^{d^-} p - \cdots - \tilde f_i p - p - \tilde e_i p - \tilde e_i^2 p - \cdots - \tilde e_i^{d^+} p$$

where $\tilde e_i^{d^+ + 1} p = 0$ and $\tilde f_i^{d^- + 1} p = 0$. If $h = \tilde e_i^{d^+} p$ ← head of i-string

then the paths in the i-string ~~have weights~~

$$\tilde f_i^{\langle \mu, \alpha_i^\vee \rangle} h - \qquad \cdots \qquad - \tilde f_i^2 h - \tilde f_i h - h$$

have weights

$$\mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i, \qquad \cdots \qquad, \mu - 2\alpha_i, \ \mu - \alpha_i, \ \mu$$

with

$$s_i \mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i .$$

Define an action of W_0 on B by setting

$s_i p$ to be the opposite of p in its i-string.

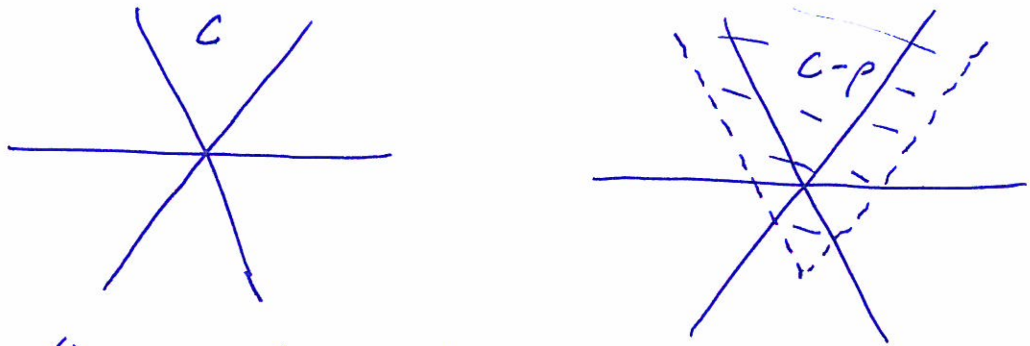
$$\tilde f_i^{\langle \mu, \alpha_i^\vee \rangle} h - \tilde f_i^{\langle \mu, \alpha_i^\vee \rangle - 1} h - \qquad \cdots \qquad - \tilde f_i^2 h - \tilde f_i h - h$$

Then
$$wt(s_i p) = s_i\, wt(p).$$

So
$$\mathrm{char}(B) = s_i\, \mathrm{char}(B) \quad \text{for } i = 1, \dots, n.$$

Since $s_1, \dots, s_n$ generate W_0 it follows that
$$\mathrm{char}(B) \in \mathbb{C}[X]^{W_0}$$

A <u>highest</u> <u>weight</u> <u>path</u> is $p \subseteq C - \rho$

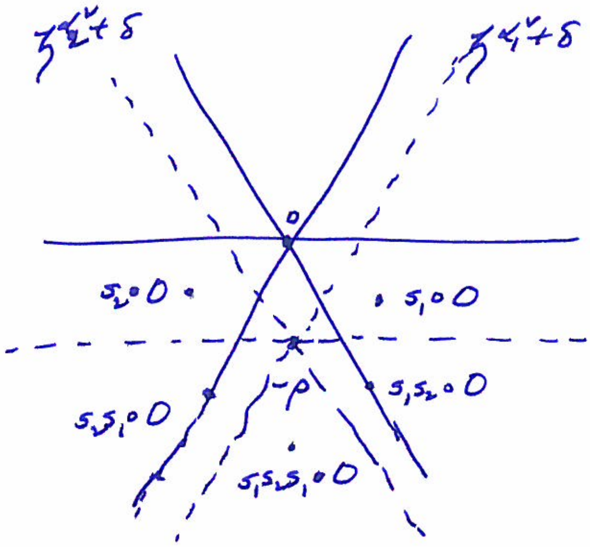


A path p is highest weight, if and only if
$$\tilde{e}_i p = 0 \quad \text{for all } i = 1, \dots, n.$$

<u>Theorem</u> Let B be a crystal. Then
$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C - \rho}} s_{wt(p)}.$$

<u>Proof</u> Define
$$s_\mu = \frac{a_{\mu+\rho}}{a_\rho}, \quad \text{for } \mu \in P.$$

Then define a new action of W_0 on P by
$$w \circ \mu = w(\mu + \rho) - \rho, \quad \text{for } \mu \in \mathfrak{h}_{\mathbb{Z}}^*, \ w \in W_0.$$



This is the <u>dot</u> <u>action</u> of W_0.

We have

$$s_{w\circ\mu} = s_{w(\mu+\rho)-\rho} = \frac{a_{w(\mu+\rho)-\rho+\rho}}{a_\rho} = \frac{a_{w(\mu+\rho)}}{a_\rho}$$

$$= \frac{\det(w)\, a_{\mu+\rho}}{a_\rho} = \det(w)\, s_\mu . \qquad (*)$$

Now

Let $\varepsilon = \sum_{w\in W_0} \det(w)\, w$ so that $\varepsilon(X^\mu) = a_\mu$.

Then

$$\mathrm{char}(B) = \frac{1}{a_\rho}\,\mathrm{char}(B)\, a_\rho = \frac{1}{a_\rho}\,\mathrm{char}(B)\,\varepsilon(\cancel{a_\rho} X^\rho)$$

$$= \frac{1}{a_\rho}\,\varepsilon(\mathrm{char}(B)\, X^\rho) = \frac{1}{a_\rho}\sum_{p\in B} \varepsilon(X^{wt(p)+\rho})$$

$$= \sum_{p\in B} s_{wt(p)}$$

The equation $(*)$ can cause some cancellation in this sum.

Let $p \in B$ such that p is not highest weight.

Let i be minimal such that p leaves $C-\rho$ by crossing $\zeta^{\alpha_i+\delta}$. Define $s_i \circ p$ to be the element of the i-string of p such that

$$\mathrm{wt}(s_i \circ p) = s_i \circ p$$

$$t - \tilde{e}_i t - \tilde{e}_i^2 t - \quad \cdots \quad \cdots \quad - \tilde{f}_i^2 h - \tilde{f}_i h - h$$

Then

$$s_{\mathrm{wt}(s_i \circ p)} = \det(s_i^{-1})\, s_{\mathrm{wt}(p)} = -s_{\mathrm{wt}(p)}$$

and

$$s_{\mathrm{wt}(s_i \circ p)} + s_{\mathrm{wt}(p)} = 0.$$

Note that $s_i \circ p$ leaves $C-\rho$ at the same place that p leaves $C-\rho$.

Thus

$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C-\rho}} s_{\mathrm{wt}(p)}. \qquad //.$$

<u>Theorem</u> Let p_λ^+ be a path, $p_\lambda^+ \subseteq C-\rho$ with $\mathrm{wt}(p_\lambda^+) = \lambda$. Let $B(\lambda)$ be the crystal generated by p_λ^+.

Then

$$\mathrm{char}(B(\lambda)) = s_\lambda.$$

<u>Theorem</u> Let B be a crystal. Let $J \subseteq \{1, \ldots, n\}$.
By ignoring the action of $\tilde{e}_i, \tilde{f}_i$ for $i \notin J$,

B is a $(W_J, \mathfrak{h}_{\mathbb{R}}^*)$-crystal,

where $W_J = \langle s_j \mid j \in J \rangle$. Let $C_J - \rho_J$ be the region on the positive side of

$\mathfrak{h}^{\alpha_j^\vee + \delta_j}$ for $j \in J$.

Then
$$\mathrm{char}(B) = \sum_{\substack{p \in B \\ p \subseteq C_J - \rho_J}} s^J_{wt(p)}$$

<u>Theorem</u> Let $\lambda, \mu \in P^+$. Let

$B(\lambda)$ and $B(\mu)$ be the irreducible crystals of highest weights λ and μ, respectively.

Then
$$B(\lambda) \otimes B(\mu) = \{p \otimes q \mid p \in B(\lambda), q \in B(\mu)\}$$
is a crystal ($p \otimes q$ is the concatenation of p and q). Then

$$\mathrm{char}(B(\lambda) \otimes B(\mu)) = \sum_{\substack{q \in B(\mu) \\ p_\lambda^+ \otimes q \subseteq C - \rho}} s_{wt(q) + \lambda}$$

<u>Proof</u> $p \otimes q \subseteq C - \rho$ only if $p = p_\lambda^+$, in which case

$s_{wt(p_\lambda^+ \otimes q)} = s_{\lambda + wt(q)}$.