Representation Theory Lecture 5 26.08.08

Review

The Lie algebra

$$sl_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a+d=0 \right\}$$

with bracket $[x,y] = xy - yx$, for $x, y \in sl_2$

is presented by generators

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

The enveloping algebra Usl_2 is generated by e, f, h with relations

$$ef = fe + h, \quad eh = he - 2e, \quad hf = fh - 2f$$

and has basis

$$\{ f^{m_1} h^{m_2} e^{m_3} \mid m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0} \}.$$

If $M = \text{span}\{m_1, \ldots, m_r\}$ and $N = \{n_1, \ldots, n_s\}$ are Usl_2-modules, then

$$M \otimes N = \text{span}\{ m_i \otimes n_j \mid 1 \leq i \leq r, 1 \leq j \leq s \}$$

is a Usl_2-module, with

$$e(m_i \otimes n_j) = em_i \otimes n_j + m_i \otimes en_j$$
$$f(m_i \otimes n_j) = fm_i \otimes n_j + m_i \otimes fn_j.$$
$$h(m_i \otimes n_j) = hm_i \otimes n_j + m_i \otimes hn_j.$$

The quantum group $U_q\mathfrak{sl}_2$

$U_q\mathfrak{sl}_2$ is generated by $E, F, K^{\pm 1}$ with relations

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F,$$

$$EF = FE + \frac{K-K^{-1}}{q-q^{-1}}.$$

The map $\Delta: U \to U \otimes U$ given by

$$\Delta(E) = E \otimes K + 1 \otimes E,$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

$$\Delta(K) = K \otimes K,$$

is a coproduct.

$U = U_q\mathfrak{sl}_2$ at $q=1$ is $U\mathfrak{sl}_2$.

$U = U_q\mathfrak{sl}_2$ has a 2-dimensional simple module

$$V = L(\square) = \mathrm{span}\{v_1, v_{-1}\} \quad \text{with}$$

$$Kv_1 = qv_1, \qquad Ev_1 = 0, \qquad Fv_1 = v_{-1},$$

$$Kv_{-1} = q^{-1}v_{-1}, \qquad Ev_{-1} = v_1, \qquad Fv_{-1} = 0.$$

So $\rho^\square: U \to \mathrm{End}(L(\square))$ has

$$\rho^\square(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad \rho^\square(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho^\square(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Computing $V \otimes V = L(\square) \otimes L(\square) = V^{\otimes 2}$

$$V^{\otimes 2} = V \otimes V = \mathrm{span}\{v_1 \otimes v_1,\ v_1 \otimes v_{-1},\ v_{-1} \otimes v_1,\ v_{-1} \otimes v_{-1}\}$$

with

$$E(v_1 \otimes v_1) = 0, \qquad E(v_{-1} \otimes v_1) = q\, v_1 \otimes v_1,$$

$$E(v_1 \otimes v_{-1}) = v_1 \otimes v_1, \qquad E(v_{-1} \otimes v_{-1}) = q^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes v_1.$$

$$K(v_1 \otimes v_1) = q^2 v_1 \otimes v_1, \quad K(v_{-1} \otimes v_{-1}) = q^{-2} v_{-1} \otimes v_{-1},$$

$$K(v_1 \otimes v_{-1}) = v_1 \otimes v_{-1}, \quad K(v_{-1} \otimes v_1) = v_{-1} \otimes v_1,$$

$$F(v_1 \otimes v_1) = v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1}, \quad F(v_{-1} \otimes v_1) = q\, v_{-1} \otimes v_{-1},$$

$$F(v_1 \otimes v_{-1}) = v_{-1} \otimes v_{-1}, \quad F(v_{-1} \otimes v_{-1}) = 0,$$

or, equivalently,

$$(\rho^\square \otimes \rho^\square)(E) = \rho^\square(E) \otimes \rho^\square(K) + \rho^\square(1) \otimes \rho^\square(E)$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} & 1 \cdot \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \\ 0 \cdot \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} & 0 \cdot \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} & \begin{matrix} q & \\ & q^{-1} \end{matrix} \\ \hline & \end{array}\right) + \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{array}\right) = \begin{pmatrix} 0 & 1 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In general, if $A = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ \vdots & & \\ b_{s1} & \cdots & b_{s1} \end{pmatrix}$

acting on $M = \mathrm{span}\{m_1, \dots, m_r\}$ and $N = \mathrm{span}\{n_1, \dots, n_s\}$ respectively then, if

$$(A \otimes B)(m_i \otimes n_j) = Am_i \otimes Bn_j,$$

then the matrix of $A \otimes B$ in the basis

$$m_1 \otimes n_1, \dots, m_1 \otimes n_s, m_2 \otimes n_1, \dots, m_2 \otimes n_s, \dots, m_r \otimes n_1, \dots, m_r \otimes n_s$$

is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1r}B \\ \vdots & & & \vdots \\ a_{r1}B & - & - & a_{rr}B \end{pmatrix}$$

<u>Decomposing $V^{\otimes 2}$</u>

$$\begin{array}{ll} v_1 \otimes v_{+1} & F(v_1 \otimes v_1) = v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1} \\ \downarrow F & \\ v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1} & F(v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1}) = [2]\, v_{-1} \otimes v_{-1}. \\ \downarrow F & \\ [2]\, v_{-1} \otimes v_{-1}. & \end{array}$$

$$v_{-1} \otimes v_1 - q v_1 \otimes v_{-1} \qquad \begin{array}{l} E(v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}) = 0 \\ F(v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}) = 0 \end{array}$$

Let

$$b_1 = v_1 \otimes v_1, \quad b_2 = v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1}, \quad b_3 = v_{-1} \otimes v_{-1}$$

$$b_4 = v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}.$$

Then

$$V^{\otimes 2} = L(\square) \otimes L(\square) = L(\square\square) \oplus L(\emptyset)$$

where

$$L(\square\square) = \mathrm{span}\{b_1, b_2, b_3\} \text{ and } L(\emptyset) = \mathrm{span}\{b_4\}$$

In the basis b_1, b_2, b_3, b_4 the matrices for the action of E, F, K on $V \otimes V$ are

$$\rho^{\square\square\oplus\emptyset}(E) = \left(\begin{array}{ccc|c} 0 & [2] & & \\ & 0 & 1 & \\ & & 0 & \\ \hline & & & 0 \end{array}\right) \quad \rho^{\square\square\oplus\emptyset}(F) = \left(\begin{array}{ccc|c} 0 & & & \\ 1 & 0 & & \\ & [2] & 0 & \\ \hline & & & 0 \end{array}\right)$$

$$\rho^{\square\square\oplus\emptyset}(K) = \left(\begin{array}{ccc|c} q^2 & & & \\ & q^0 & & \\ & & q^{-2} & \\ \hline & & & 1 \end{array}\right)$$

<u>Decomposing</u> $V^{\otimes 3} = L(\square) \otimes L(\square) \otimes L(\square)$.

$$L(\square) \otimes L(\square) \otimes L(\square) = \mathrm{span}\left\{\begin{array}{l} v_1 \otimes v_1 \otimes v_1, \ v_1 \otimes v_1 \otimes v_{-1} \\ v_1 \otimes v_{-1} \otimes v_1, \ v_1 \otimes v_{-1} \otimes v_{-1} \\ v_{-1} \otimes v_1 \otimes v_1, \ v_{-1} \otimes v_1 \otimes v_{-1} \\ v_{-1} \otimes v_{-1} \otimes v_1, \ v_{-1} \otimes v_{-1} \otimes v_{-1} \end{array}\right\}$$

Another basis of $V^{\otimes 3}$ is

$$\{b_1\otimes v_1,\ b_1\otimes v_{-1},\ b_2\otimes v_1,\ b_2\otimes v_{-1},$$
$$b_3\otimes v_1,\ b_3\otimes v_{-1},\ b_4\otimes v_1,\ b_4\otimes v_{-1}\},$$

i.e. $V^{\otimes 3} = (L(\Box\Box)\oplus L(\phi))\otimes V = (L(\Box\Box)\otimes V)\oplus(L(\phi)\otimes V)$

$L(\phi)\otimes V = \text{span}\{b_4\otimes v_1,\ b_4\otimes v_{-1}\}$ with

$$E(b_4\otimes v_1)=0,\qquad F(b_4\otimes v_1)=b_4\otimes v_{-1},\qquad K(b_4\otimes v_1)=q\,b_4\otimes v_1$$
$$E(b_4\otimes v_{-1})=b_4\otimes v_1,\qquad F(b_4\otimes v_{-1})=0,\qquad K(b_4\otimes v_{-1})=q^{-1}b_4\otimes v_{-1},$$

So $\quad L(\phi)\otimes V \simeq L(\Box)$

$$b_4\otimes v_1 \longmapsto v_1$$
$$b_4\otimes v_{-1}\longmapsto v_{-1}$$

Then $L(\Box\Box)\otimes V = \text{span}\left\{\begin{matrix} b_1\otimes v_1,\ b_1\otimes v_{-1},\ b_2\otimes v_1,\ b_2\otimes v_{-1}, \\ b_3\otimes v_1,\ b_3\otimes v_{-1}\end{matrix}\right\}$

with

$$b_1\otimes v_1 \qquad\qquad F(b_1\otimes v_1) = b_2\otimes v_1 + q^{-2}b_1\otimes v_{-1}$$
$$\downarrow F$$
$$b_2\otimes v_1 + q^{-2}b_1\otimes v_{-1}$$
$$\downarrow F$$
$$[2]b_3\otimes v_1 + b_2\otimes v_{-1} + q^{-2}b_2\otimes v_{-1} + q^{-4}\cancel{b_2\otimes 0} = [2](b_3\otimes v_1 + q^{-1}b_2\otimes v_{-1})$$
$$\downarrow F$$
$$0+[2]q^{2}b_3\otimes v_{-1} + [2]b_3\otimes v_{-1} + 0 + q^{-2}[2]b_3\otimes v_{-1} + 0$$
$$= [2][3]\,b_3\otimes v_{-1}$$

So, if

$$c_1 = b_1 \otimes v_1, \quad c_2 = b_2 \otimes v_1 + q^{-2} b_1 \otimes v_{-1}$$

$$c_3 = b_3 \otimes v_1 + q^{-1} b_2 \otimes v_{-1}, \quad c_4 = b_3 \otimes v_{-1}$$

then the action of F on

$$L(\square\square\square) = \mathrm{span}\{c_1, c_2, c_3, c_4\} \text{ is given by}$$

$$\rho^{\square\square\square}(F) = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & [2] & 0 & \\ & & [3] & 0 \end{pmatrix} \quad \text{and}$$

$$\rho^{\square\square\square}(E) = \begin{pmatrix} 0 & 1 & & \\ & 0 & [2] & \\ & & 0 & [3] \\ & & & 0 \end{pmatrix}, \quad \rho^{\square\square\square}(K) = \begin{pmatrix} q^3 & & & \\ & q^1 & & \\ & & q^{-1} & \\ & & & q^{-3} \end{pmatrix}$$

Note that

$$\begin{array}{c} 0 \\ \uparrow E \\ b_2 \otimes v_1 - q b_1 \otimes v_{-1} \\ F \downarrow \\ [2] b_3 \otimes v_1 + b_2 \otimes v_{-1} - q[2] b_2 \otimes v_{-1} \\ \qquad\qquad\qquad = [2] b_3 \otimes v_{-1} - q^2 b_2 \otimes v_{-1} \\ F \downarrow \\ [2] q^2 b_3 \otimes v_{-1} - q^2 [2] b_3 \otimes v_{-1} = 0 \end{array}$$

So that, if

$$c_5 = b_2 \otimes v_1 - q b_1 \otimes v_{-1}$$

$$c_6 = [2] b_3 \otimes v_{-1} - q^2 b_2 \otimes v_{-1}$$

and $L(\square) = \mathrm{span}\{c_5, c_6\}$

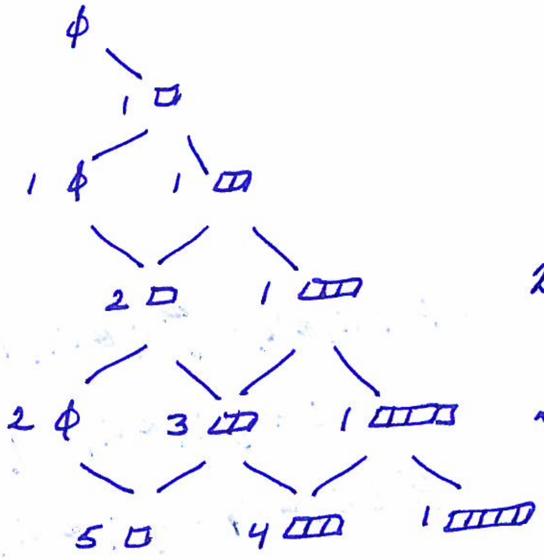
then $\tilde{L}(\square) \simeq L(\square)$

So

$$V^{\otimes 3} = L(\square) \otimes L(\square) \otimes L(\square) \cong \big(L(\square\square) \oplus L(\phi)\big) \otimes L(\square)$$

$$= \big(L(\square\square) \otimes V\big) \oplus \big(L(\phi) \otimes V\big)$$

$$\simeq L(\square\square\square) \oplus L(\square) \oplus L(\square)$$



$1 \cdot 2 = 2$

$1 \cdot 1 + 1 \cdot 3 = 4$

$2 \cdot 2 + 1 \cdot 4 = 8$

$2 \cdot 1 + 3 \cdot 3 + 1 \cdot 5 = 16$

$5 \cdot 2 + 4 \cdot 4 + 1 \cdot 6 = 32$

<u>What is the connection between TL_k and $V^{\otimes k}$ for $U_q\mathfrak{sl}_2$?</u>

Define an action of $TL_2 = \mathrm{span}\{\mathbb{1}, \stackrel{\cup}{\cap}\}$ on

$$V^{\otimes 2} = \mathrm{span}\{v_1 \otimes v_1,\ v_1 \otimes v_{-1},\ v_{-1} \otimes v_1,\ v_{-1} \otimes v_{-1}\}$$

by

$$\stackrel{\cup}{\cap}(v_1 \otimes v_1) = 0, \qquad \stackrel{\cup}{\cap}(v_{-1} \otimes v_{-1}) = 0$$

$$\stackrel{\cup}{\cap}(v_1 \otimes v_{-1}) = q v_1 \otimes v_{-1} - v_{-1} \otimes v_1$$

$$\stackrel{\cup}{\cap}(v_{-1} \otimes v_1) = q^{-1} v_{-1} \otimes v_1 - v_1 \otimes v_{-1}$$

In matrices we have

$$\rho^{\otimes 2}\left(\overset{\cup}{\cap}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ Note that

$$\left(\rho^{\otimes 2}\left(\overset{\cup}{\cap}\right)\right)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^2+1 & -(q+q^{-1}) & 0 \\ 0 & -(q+q^{-1}) & 1-q^{-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = [2]\,\rho^{\otimes 2}\left(\overset{\cup}{\cap}\right).$$

so that this is an action of TL_2 on $V^{\otimes 2}$.

The TL_2 action <u>commutes</u> with the $U_q sl_2$ action on $V^{\otimes 2}$, i.e.

$$\rho^{\otimes 2}(TL_2) \subseteq \mathrm{End}_U(V^{\otimes 2})$$

The <u>Temperley-Lieb</u> <u>algebra</u> TL_k is generated by

$$e_j = \begin{array}{cccccccccc} 1 & 2 & \cdots & & & j & j+1 & \cdots & & k \\ | & | & | & | & | & \overset{\cup}{\cap} & & | & | \; | & | \end{array}, \qquad 1 \le j \le k-1.$$

Define an action of TL_k on

$$V^{\otimes k} = \mathrm{span}\{ v_{i_1} \otimes \cdots \otimes v_{i_k} \mid i_1, \ldots, i_k \in \{\pm 1\} \}$$

by

$$e_j (v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes \overset{\cup}{\cap}(v_{i_j} \otimes v_{i_{j+1}}) \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_k}$$

<u>Claim</u> (a) This defines a TL_k-action on $V^{\otimes k}$

(b) This TL_k-action <u>commutes</u> with the $U_q\mathfrak{sl}_2$-action on $V^{\otimes k}$.

Let A be an algebra and let M be a semisimple A module,

$$M = \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus m_\lambda}.$$

Let $Z = \mathrm{End}_A(M)$. Then

$$Z = \bigoplus_{\lambda \in \hat{M}} M_{m_\lambda}(\mathbb{C}), \quad \text{and} \quad M \cong \bigoplus_{\lambda \in \hat{M}} A^\lambda \otimes Z^\lambda$$

as an (A, Z) bimodule, where $\hat{M} \subseteq \hat{A}$ is an index set for the simple A-modules appearing in M.

<u>Proof</u>

$$Z = \mathrm{Hom}_A(M, M) = \mathrm{Hom}_A\Big(\bigoplus_{\lambda \in \hat{M}} \bigoplus_{i=1}^{m_\lambda} A_i^\lambda, \bigoplus_{\mu \in \hat{M}} \bigoplus_{j=1}^{m_\lambda} A_j^\mu\Big)$$

$$= \bigoplus_{\lambda \in \hat{M}} \bigoplus_{i,j=1}^{m_\lambda} \mathrm{Hom}_A(A_i^\lambda, A_j^\lambda),$$

by Schur's Lemma. Hence

$$Z = \mathrm{span}\{e_{ij}^\lambda \mid \lambda \in \hat{M}, 1 \le i,j \le m_\lambda\} \quad \text{where } e_{ij}^\lambda : A_i^\lambda \to A_j^\lambda$$

(choose e_{ii}^λ so that $(e_{ii}^\lambda)^2 = e_{ii}^\lambda$ and e_{ij}^λ and e_{ji}^λ so that $e_{ij}^\lambda e_{ji}^\lambda = e_{ii}^\lambda$).