# REPRESENTATION THEORY

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Abstract. Notes from Arun Ram's 2008 course at the University of Melbourne.

## 4. Week 4

**Question.** What does $_q$ in question 2 of the homework mean?

**Answer.** $\mathcal{U}_q(\mathfrak{sl}_2)$ has generators $E$, $F$, and $K$ satisfying certain relations, I think I specified them in the assignment.

**Question.** Is there a nice way to see that the (half) twist generates the center of the braid group?

**Answer.** This is related to other questions, like what are the conjugacy classes of the braid group? And the word problem: How do you tell is one braid is conjagate to another? Garside and Deligne solve this, and the center problem, all at once.

Today's lecture is about getting you the tools you need to do the homework.

### 4.1. Irreducible representations of $H_k$.

**Recall.** The *Iwahori-Hecke algebra* $H_k$ is generated by $T_1, \ldots, T_{k-1}$ (PIC) with relations $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ and $T_i = T_i^{-1} = q - q^{-1}$

**Remark.** If $q = 1$ then $H_k = \mathbb{C}S_k$.

Your goal is to find the irreducible representations of the Hecke algebra.

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*Date*: August 20, 2008.

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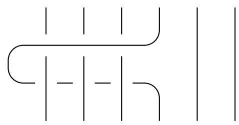
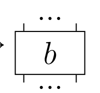
**Recall.** $y^{\varepsilon_i^\vee} =$  $= T_{i-1}T_{i-2}\cdots T_2T_1^2T_2\cdots T_{i-1}$. These are good elements because they satisfy $y^{\varepsilon_i^\vee}y^{\varepsilon_j^\vee} = y^{\varepsilon_j^\vee}y^{\varepsilon_i^\vee}$

We will use $\mathrm{Res}_{H_{k-1}}^{H_k}$ and $\mathrm{Ind}_{H_{k-1}}^{H_k}$ to study $H_1 \subset H_2 \subset H_3 \subset \cdots$ where

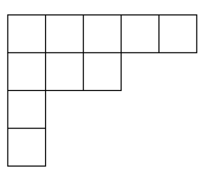
$$H_k \hookrightarrow H_{k+1}$$

$b \mapsto$ 

By counting dimensions we can see that $H_1 \simeq M_1(\mathbb{C})$ has one irrep, $H_2 \simeq M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ has two irreps, $H_3 \simeq M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ or $H_3 \simeq M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$.

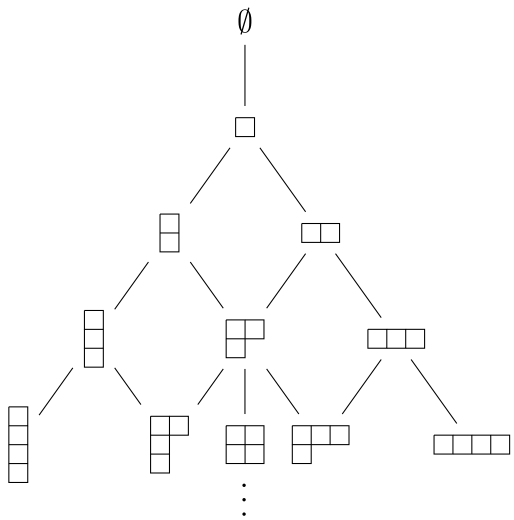
Let me tell you what the irreducible representations of $H_k$ are; your job will be to prove it's correct.

**Definition.** A *partition* is a collection of boxes in a corner. We write a partition as $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ where $\lambda_i$ = number of boxes in row $i$.

**Example.** $\lambda =$  $= (5, 3, 1, 1)$

Now, $\hat{H}_k = \{\text{partitions } \lambda \text{ with } k \text{ boxes}\}$. (This hat should make you think this set is in one-to-one correspondence with ireducible $H_k$ modules.)

**Definition.** The *Bratteli diagram* of $H_1 \subset H_2 \subset H_3$ has vertices on level $k$ corresponding to $\lambda \in \hat{H}_k$. The vertices $\lambda$ and $\mu$ are connected by an edge if $\mu$ is obtained from $\lambda$ by adding a box.



So assuming that $\hat{H}_k$ is in 1-1 correspondence with irreducible $H_k$-modules via $\lambda \mapsto H_k^\lambda$, the Bratteli diagram says

$$\mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\mu) = \bigoplus_{\substack{\lambda \in \hat{H}_{k-1}, \\ \mu/\lambda = \square}} H_{k-1}^\lambda$$

**Example.**

$$\mathrm{Res}^{H_4}_{H_3}(H_4^{\yng(2,1,1)}) = H_3^{\yng(1,1,1)} \oplus H_3^{\yng(2,1)}$$

Since $\mathrm{Hom}_{H_k}(\mathrm{Ind}^{H_k}_{H_{k-1}}(H_{k-1}^\lambda), H_k^\mu) \simeq \mathrm{Hom}_{H_{k-1}}(H_{-1}^\lambda, \mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\mu))$, by Schur's lemma, we get

$$\mathrm{Ind}^{H_k}_{H_{k-1}}(H_{k-1}^\lambda) = \bigoplus_{\substack{\mu \in \hat{H}_k, \\ \mu/\lambda = \square}} H_k^\mu$$

But we haven't said what $H_k^\lambda$ is. Can we build $H_k^\lambda$?

**Question.** Well uh uh what is $\dim(H_k^\lambda)$?

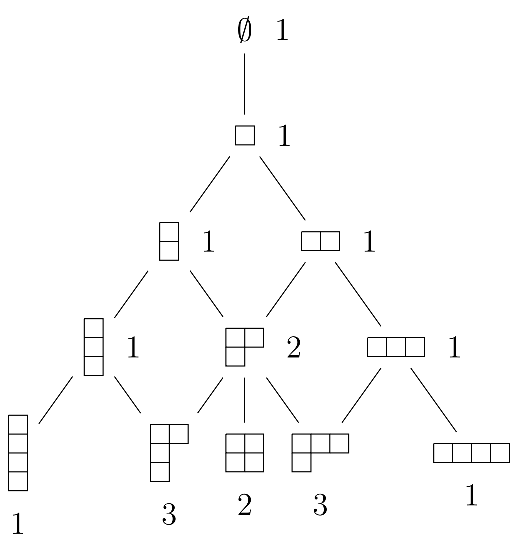
As vector spaces,

$$\begin{aligned} H_5^{\yng(2,2,1)} &= H_4^{\yng(2,2)} + H_4^{\yng(2,1,1)} = H_3^{\yng(2,1)} + H_3^{\yng(1,1,1)} + H_3^{\yng(2,1)} \\ &= H_2^{\yng(1,1)} + H_2^{\yng(2)} + H_2^{\yng(2)} + H_2^{\yng(1,1)} + H_2^{\yng(2)} \\ &= H_1^{\yng(1)} + H_1^{\yng(1)} + H_1^{\yng(1)} + H_1^{\yng(1)} + H_1^{\yng(1)} \end{aligned}$$

so $\dim H_5^{\yng(2,2,1)} = 5$.

By tracing where each box in the final lines comes from, we end up getting a path in the Bratteli diagram. That is, $\dim H_k^\lambda =$ number of paths from $\emptyset$ to $\lambda$ in the Bratteli diagram.

To count paths on the Bratteli diagram, just do a Pascal triangle type thing.



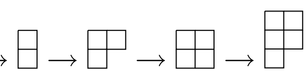
For example, $H_4 \simeq M_1(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_1(\mathbb{C})$.

**Definition.** A *standard tableau of shape* $\lambda$ is a filling of the boxes with $1, 2, \ldots k$ such that the rows increase left to right and the columns increase top to bottom.

**Example.** $\lambda = (2,2,1)$ has standard tableau

$$\begin{array}{ll} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}, \quad \begin{array}{ll} 1 & 3 \\ 2 & 5 \\ 4 & \end{array}, \quad \begin{array}{ll} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}, \quad \begin{array}{ll} 1 & 2 \\ 3 & 5 \\ 4 & \end{array}, \quad \begin{array}{ll} 1 & 2 \\ 3 & 4 \\ 5 & \end{array},$$

It should be obvious that these guys are the same as paths. There is a bijection beween standard tableaux of shape $\lambda$ and paths from $\emptyset$ to $\lambda$.

**Example.** $\begin{array}{ll} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}$ corresponds to $\emptyset \to \square \to$ 

Note: $k! = \sum_{\lambda \vdash k} (\dim H_k^\lambda)^2 = \sum_{\lambda \vdash k} (\#\text{ of standard tableau of shape } \lambda)^2$

As a vector space $H_k^\lambda$ has basis $\{v_T | T \text{ is a standard tableau of shape } \lambda\}$.

**Question.** $H_k$ acts on $H_k^\lambda$ how? $y^{\varepsilon_i^\vee} v_T = ?$

Why define the action of $y^{\varepsilon_i^\vee}$ instead of $T_i$? Because the $y$'s commute with each other, so we can look for a basis in which they're all diagonal.

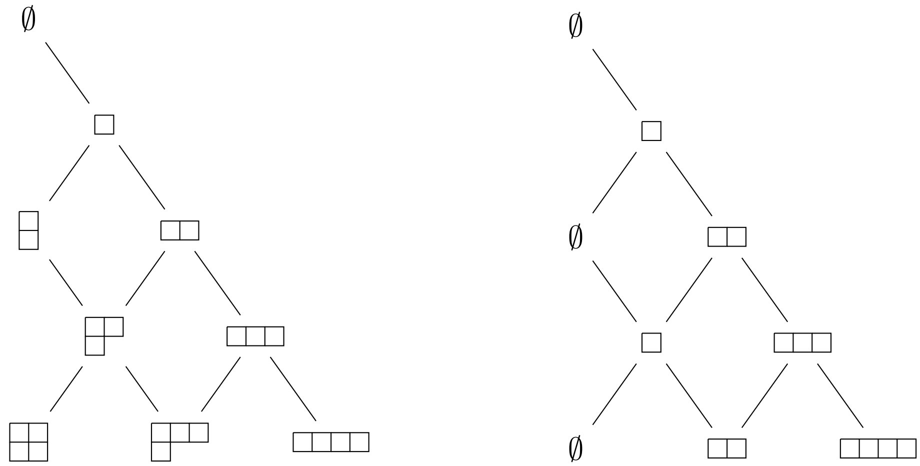
**Theorem 4.1.** *$H_k$ acts on $H_k^\lambda$ by $y^{\varepsilon_i^\vee} v_T = q^{c(T(i))} v_T$. You can unravel this and get the much nastier formulation*

$$T_i v_T = \frac{q - q^{-1}}{1 - q^{2(c(Y(i)) - c(T(i+1)))}} v_T + (q^{-1} + \frac{q - q^{-1}}{1 - q^{2(c(T(i)) - c(T(i+1)))}}) v_{s_i T}.$$

*Here $T(i)$ is the box containing $i$ in $T$ and the content $c$ of a box $b$ is $c(b) = s - r$, where $b$ is in column $r$, row $s$. Also $s_i T$ is defined to be $T$ except $i$ and $i+1$ are switched, and $v_{s_i T} = 0$ if $s_i T$ is not standard.*

4.2. **Irreducible representations of Temperley-Lieb.** There is a surjective map $H_k \to TL_k$ via $T_i - q \mapsto e_i$. So every $TL_k$-module is an $H_k$-module.

For $TL_1 \subset TL_2 \subset TL_3 \subset \cdots$, the Bratteli diagram is



(You can get the second diagram from the first by deleting 2-row columns from all the tableau). Now you have (more than enough) tools to do problem 1 on the homework.

4.3. **Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$.** There's a lie algebra called $\mathfrak{sl}_2$. Circa 1985, $\mathfrak{sl}_2$-irreducible modules were written down by R. Block. $\mathfrak{sl}_3$ is thought to be impossible. Arun thinks $\mathcal{U}_q(\mathfrak{sl}_2)$ is not done, and also that it's not very hard. So he assigned it as homework.

Note: the relation between $\mathfrak{sl}_2$ and $\mathcal{U}_q(\mathfrak{sl}_2)$ is the same as the relation between $H_k$ and $\mathbb{C}S_k$: set q=1 to pass from the first to the second.

**Definition.** A *Lie algebra* is a vector space $\mathfrak{g}$ with a bracket

$$[,] : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$$

such that
(a) $[x, y] = -[y, x]$ for $x, y \in \mathfrak{g}$;
(b) $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ for $x, y, z \in \mathfrak{g}$ (this is called the Jacobi identity).

Note that a Lie algebra is not an algebra – Lie is not an adjective – maybe we should write it Liealgebra. But seriously, this more than just a grammatical problem, because we don't know how to talk about representations of anything but algebras.

**Definition.** The *enveloping algebra* of $\mathfrak{g}$ is the algebra $\mathcal{U}(\mathfrak{g})$ generated by the vector space $\mathfrak{g}$ with relations $xy = yx + [x, y]$ for $x, y \in \mathfrak{g}$

**Definition.** A $\mathfrak{g}$-*module* is a $\mathcal{U}(\mathfrak{g})$-module.

**Definition.** $\mathfrak{sl}_2 = \{x \in M_2(\mathbb{C}) | \mathrm{tr}(x) = 0\} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a + d = 0\}$ with

$$[x, y] = xy - yx$$

where the product on the RHS is matrix multiplication.

**Proposition 4.2.** $\mathfrak{sl}_2$ *is generated by*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

*with relations* $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

Then $\mathcal{U}(\mathfrak{sl}_2)$ is the algebra generated by $e, f, h$ with relations

$$ef = fe + h, \; eh = he - 2e, \; hf = fh - 2f.$$

Thus $\mathcal{U}(\mathfrak{sl}_2)$ has basis $\{f^{m_1} h^{m_2} e^{m_3} | m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}\}$. So $\mathcal{U}(\mathfrak{sl}_2)$ is sort of like the polynomial ring $\mathbb{C}[\epsilon, \phi, \eta]$ (which has relations $\epsilon\phi = \phi\epsilon$, $\epsilon\eta = \eta\epsilon$, $\eta\phi = \phi\eta$.)

One problem with $\mathcal{U}(\mathfrak{sl}_2)$ is that it's infinite dimensional, so we can't use Artin-Wedderburn. This is why finding its modules was considered a hard problem.

Let's build the modules.

**Definition.** Let $L(\square) = \mathrm{span}\,\{v_1, v_{-1}\}$ with

$$\begin{aligned} ev_1 &= 0, & fv_1 &= v_{-1}, & hv_1 &= v_,1 \\ ev_{-1} &= v_1, & fv_{-1} &= 0, & hv_{-1} &= -v_{-1} \end{aligned}$$

Note that this is just the representation we get from writing $e$, $f$ and $h$ as 2-by-2 matrices.

How can we build more modules? Luckily $\mathcal{U} = \mathcal{U}(\mathfrak{sl}_2)$ is a Hopf algebra! What does this mean? Let $M$ and $N$ be $\mathcal{U}$-modules. $M$ has basis $\{m_1, \ldots, m_r\}$, $N$ has basis $\{n_1, \ldots, n_s\}$. $M \otimes N$ has basis $\{m_i \otimes n_j\}$ and $\dim M \otimes N = rs$. Saying $\mathcal{U}$ is a Hopf algebra means that $\mathcal{U}$ comes with a map $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$, called the *coproduct*, which makes $\mathcal{U}$ act on $M \otimes N$. Define $\Delta$ by

$$\Delta(e) = e \otimes 1 + 1 \otimes e, \Delta(f) = f \otimes 1 + 1 \otimes f, \Delta(h) = h \otimes 1 + 1 \otimes h$$

Now $L(\square) \otimes L(\square)$ has basis $\{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$. So, for example,

$$\begin{aligned} e(v_1 \otimes v_1) &= \Delta(e)(v_1 \otimes v_1) = (e \otimes 1 + 1 \otimes e)(v_1 \otimes v_1) \\ &= ev_1 \otimes v_1 + v_1 \otimes ev_1 = 0 \end{aligned}$$

A more interesting example is

$$\begin{aligned} f(v_1 \otimes v_1) &= \Delta(f)(v_1 \otimes v_1) = (f \otimes 1 + 1 \otimes f)(v_1 \otimes v_1) \\ &= fv_1 \otimes v_1 + v_1 \otimes fv_1 = v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \end{aligned}$$

and

$$f^2(v_1 \otimes v_1) = v_{-1} \otimes v_{-1} + v_{-1} \otimes v_{-1} = 2v_{-1} \otimes v_{-1}.$$

Let $v_2 = v_1 \otimes v_1$, $v_0 = fv_2$, $2v_{-2} = f^2 v_2$ $(v_{-2} = v_{-1} \otimes v_{-1})$ and $v^0 = v_1 \times v_{-1} - v_{-1} \otimes v_1$. Then $ev^0 = 0$, $fv^0 = 0$.

**Definition.** Let $L(\square\square) = \operatorname{span}\{v_2, v_0, v_{-2}\}$, $L(\emptyset) = \operatorname{span}\{v^0\}$.

Then $L(\square) \otimes L(\square) = L(\square\square) + L(\emptyset)$.

Define $\rho^{\square\square} : \mathcal{U}(\mathfrak{sl}_2) \to \operatorname{End}(L(\square\square))$ by

$$e \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, f \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, h \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and define $\rho^{\emptyset} : \mathcal{U}(\mathfrak{sl}_2) \to \operatorname{End}(L(\emptyset))$ by

$$e \mapsto 0, f \mapsto 0, h \mapsto 0.$$

Did I give you the coproduct on $\mathcal{U}_q(\mathfrak{sl}_2)$? Maybe not. OK:

**Definition.** The coproduct $\Delta : \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2)$ is defined by

$$\begin{aligned} \Delta(E) &= E \otimes 1 + K \otimes E \\ \Delta(F) &= F \otimes K^{-1} + 1 \otimes F \\ \Delta(K) &= K \otimes K \end{aligned}$$