REPRESENTATION THEORY

EMILY PETERS

Abstract. Notes from Arun Ram's 2008 course at the University of Melbourne.

3. Week 3

Question. Why is $(q+q^{-1})$ the q-analogue of 2?

Answer. If $q=1$ then this is 2. More generally, we can define

$$[n] = \frac{q^n-1}{q-1} = 1+q+\cdots q^{n-1}$$

And we can also define

$$[n]! = [n][n-1]\cdots[2][1] \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

Note that (perhaps surprisingly) this last quantity is a genuine polynomial, not just a quotient of polynomials.

In analogy to the binomial theorem, we have: If $xy = qyx$ then we have $(x+y)^n = \sum \begin{bmatrix} n \\ k \end{bmatrix} x^n y^{n-k}$.

3.1. Heading towards Artin-Wedderburn Theorem.

Theorem 3.1. *Let A be a finite dimensional algebra such that the trace of the regular representation is nondegenerate. Then A is isomorphic to a direct sum of matrix algebras. More precisely, $A = \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C})$.*

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Send comments and corrections to E.Peters@ms.unimelb.edu.au.

We picture this as

$$\text{RHS} = \left\{ \begin{pmatrix} \boxed{*}\}d_\lambda & & 0 \\ & \boxed{*}\}d_\mu & \\ 0 & & \boxed{*} \end{pmatrix} \right\}.$$

$\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$ has basis $\{E^\lambda_{i,j} | \lambda \in \hat{A}, 1 \leq i, j \leq d_\lambda\}$. $E^\lambda_{i,j}$ has 1 in ith row, jth column of λth block and all other entries 0. Meanwhile A has basis $B = \{b\}$. *In practice, Artin-Wedderburn says we can change basis from B to $E^\lambda_{i,j}$*. How do we do this?

Idea of Proof. A is an A-module (A acts on A by left multiplication).

Maschke says we can decompose A into simple modules:

$$A = \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{n_\lambda}$$

where $\hat{A}$ is an index set for the simples and n_λ is the number of times A^λ appears in A.

We have a map

$$\begin{aligned} \rho^\lambda : A &\to \text{End}(A^\lambda) = M_{d_\lambda}(\mathbb{C}) \\ a &\mapsto \rho^\lambda(a) \end{aligned}$$

where $\rho^\lambda(a)$ is the action of a on A^λ.

$M_d(\mathbb{C})$ has $\mathbb{C}^d = \text{span}\,\{e_1, \ldots, e_d\}$, where e_i is a column vector with a 1 in the ith row, as a module, and $M_d(\mathbb{C}) = (\mathbb{C}^d)^d$.

Homework. $\mathbb{C}^d$ is a simple $M_d(\mathbb{C})$ module! It's the only one!

$\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$ has simple modules $A^\lambda = \mathbb{C}^{d_\lambda} = \text{span}\left\{e^\lambda_i | 1 \leq i \leq d_\lambda\right\}$ with $E^\mu_{i,j} e^\lambda_r = \delta_{\mu,\lambda}\delta_{j,r} e^\lambda_i$.

Back to Artin-Wedderburn: To find the isomorphism from A to $\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$, we need to change basis from $B = \{b\}$ to $\{E^\lambda_{i,j} | \lambda \in \hat{A}, 1 \leq i, j \leq d_\lambda\}$. We found $\hat{A}$ and the d_λ by decomposing A as an A-module (using

Maschke). We have

$$\rho^\lambda : A \to \mathrm{End}(A^\lambda) = M_{d_\lambda}(\mathbb{C}) \quad \text{so} \quad \rho := \bigoplus_{\lambda \in \hat{A}} \rho^\lambda : A \to \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$$

$$a \mapsto \rho^\lambda(a) \qquad\qquad a \mapsto \begin{pmatrix} \rho^\lambda(a) & & \\ & \rho^\mu(a) & \\ & & \rho^\nu(a) \end{pmatrix}$$

If $b \in B$ then

$$\rho(b) = \sum_{\lambda \in \hat{A}} \sum_{i,j=1}^{d_\lambda} \rho^\lambda(b)_{i,j} E^\lambda_{i,j}.$$

But ρ is injective (since A acts faithfully on itself by left multiplicaiton), so we identify b with $\rho(b)$:

$$b = \sum_{\lambda \in \hat{A}} \sum_{i,j=1}^{d_\lambda} \rho^\lambda(b)_{i,j} E^\lambda_{i,j}.$$

Now we want

$$E^\lambda_{i,j} = \sum_{b \in B} ??b$$

in order to prove this is an isomorphism.

(Fourier Inversion – noncommutative)

$$E^\lambda_{i,j} = \sum_{b \in B} \rho^\lambda(b^*)_{j,i} b, \tag{1}$$

where $\{b^*\}$ is the dual basis to B with respect to $\langle , \rangle$ defined by $\langle x, y \rangle = t(xy)$, where t is the trace of the regular representation.

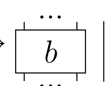
Why does this work? The point is that (1) does not depend on the choice of B. And (1) is trivial if $B = \{E^\lambda_{i,j} | \lambda \in \hat{A}, 1 \leq i, j \leq d_\lambda\}$.

Homework. Work this out and make this proof more formal

$\square$

3.2. Towers of algebras and some families of algebras. For all k we have

$$TL_k \hookrightarrow TL_{k+1}$$

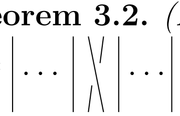
$b \mapsto$ 

which are injective algebras homomorphisms. Then

$$TL_1 \subset TL_2 \subset TL_3 \subset \cdots$$

is a "tower of algebras."

Definition. The *braid group* B_k is the group of braids on k strands with product $b_1 b_2 = b_1$ stacked on top of b_2.

Theorem 3.2. *(Artin) The braid group B_k is presented by generators* $T_i =$  $, 1 \leq i \leq k-1$ *with relations* $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

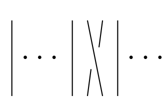
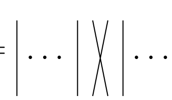
Definition. Let G be a group. The *group algebra* of G is the vector space $\mathbb{C}G$ with basis G with product determined by the product in G (and distributive laws). Sometimes this product is called convolution.

Definition. A G-module is a $\mathbb{C}G$ module.

Definition. The *symmetric group* S_k is given by generators $s_1, \ldots, s_k$ and relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i = s_i^{-1}$.

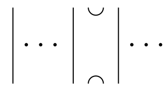
Now we have two more towers of algebras: $\mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \mathbb{C}B_3 \subset \cdots$ and $\mathbb{C}S_1 \subset \mathbb{C}S_2 \subset \mathbb{C}S_3 \subset \cdots$.

We also have a surjective map $B_k \twoheadrightarrow S_k$ given by

$T_i =$  $\mapsto s_i =$ 

Definition. The *Iwahori-Hecke algebra* H_k is the quotient of $\mathbb{C}B_k$ by $T_i = T_i^{-1} + (q - q^{-1})$, for $1 \leq i \leq k-1$.

Remark. If $q = 1$, then $H_k = \mathbb{C}S_k$. The Gram matrix of the form $\langle , \rangle$ for H_k is a matrix of polynomials. If Artin-Wedderburn works for S_k (ie, the hypothesis is satisfied) then it works for H_k – if the polynomial in q which is the determinant of the Gram matrix is non-zero for $q = 1$ then it's non-zero as a polynomial.

Let $e_i = T_i - q$ in H_k. (Recall that in TL_k, $e_i =$  and $e_i^2 = [2]e_i$, $e_ie_{i\pm1}e_i = e_i$.)

Homework. Assuming that $T_i = T_i^{-1} + (q - q^{-1})$, as it is in H_k, then $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ is equivalent to $e_ie_{i+}e_i - e_{i+1}e_ie_{i+1} = e_i - e_{i+1}$ and $T_i = T_i^{-1} + (q + q^{-1})$ is equivalent to $e_i^2 = [2]e_i$.

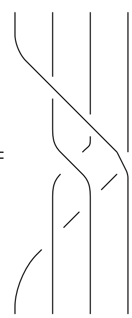
Proposition 3.3. *H_k is presented by generators $e_1, \ldots, e_{k-1}$ and relations $e_i^2 = [2]e_i$ and $e_ie_{i+}e_i - e_{i+1}e_ie_{i+1} = e_i - e_{i+1}$. So*

$$\begin{aligned} H_k &\twoheadrightarrow TL_k \\ e_i &\mapsto e_i \end{aligned}$$

is a surjective homomorphism.

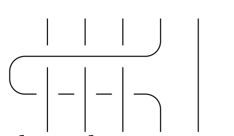
So the picture thus far is:

$$\begin{array}{ccccc} & & \mathbb{C}B_k & & \\ & & \downarrow & & \\ \mathbb{C}S_k & \underset{q=1}{\leftsquigarrow} & H_k & \twoheadrightarrow & TL_k \end{array}$$

Definition. $T_{w_0} =$ 

Remark. $T_{w_0}^2$ is a full rotation of all strands. By drawing pictures, it's not hard to convince yourself that $T_{w_0}^2 T_i T_{w_0}^2 = T_i$, so $T_{w_0}^2 \in Z(B_k)$.

Theorem 3.4. *(Arnold or Artin, Garside-Deligne) $Z(B_k)$ is generated by $T_{w_0}^2$.*

Definition. Let $y^{\varepsilon_i^\vee} =$ , where the ith strand is pulled across and then under the others.

Then $T_{w_0}^2 = y^{\varepsilon_1^\vee}y^{\varepsilon_2^\vee}\cdots y^{\varepsilon_k^\vee}$ and $y^{\varepsilon_i^\vee}y^{\varepsilon_j^\vee} = y^{\varepsilon_j^\vee}y^{\varepsilon_i^\vee}$. So, $\mathbb{C}[y^{\varepsilon_1^\vee}, \ldots, y^{\varepsilon_k^\vee}] \subset \mathbb{C}B_k$.

The game for studying the towers is to, inductively, find eigenvalues (and eigenvectors) for $T_{w_0}^2$ and $y^{\epsilon_1^\vee}, \ldots, y^{\epsilon_k^\vee}$ as operators on modules.

3.3. Tools for next week.

Definition (Pullback functors)**.** Let $\phi : A \to R$ be an algebra homomorphism. Then we get

$$\phi^* : R\text{-modules} \to A\text{-modules}$$
$$M \mapsto M$$

where A acts on M by $a \cdot m = \phi(a)m$ for $a \in A$.

If $\phi : A \hookrightarrow R$ (A is a subalgebra of R), then $\phi^*(M)$ is M with A-action from $A \subset R$ (a forgetful functor). We write $\mathrm{Res}_A^R(M)$ (an A-module).

Definition (Adjoint functors)**.** Say $F : \{R\text{-modules}\} \to \{A\text{-modules}\}$ is a functor. The *adjoint functor* is $F^\vee : \{A\text{-modules}\} \to \{R\text{-modules}\}$ determined by $\mathrm{Hom}_R(F^\vee M, N) = \mathrm{Hom}_A(M, FN)$.

Example. If $F = \mathrm{Res}_A^R$ then $F^\vee = \mathrm{Ind}_A^R$.

Given some tower, for instance $H_1 \subset H_2 \subset H_3 \subset \cdots$, Res moves us down the tower and Ind modes us up the tower.