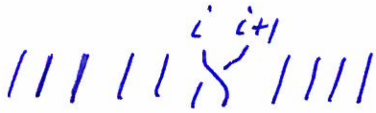
Representation Theory 2008, Lecture 4   8/19/2008

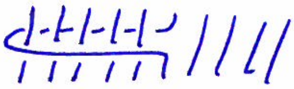
The <u>Iwahori-Hecke algebra</u> $H_k(q)$ has generators

$$T_i = $$  $$, \qquad 1 \le i \le k+1,$$

and relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{and} \quad T_i - T_i^{-1} = (q - q^{-1}).$$

If $q=1$ then $H_k(q)$ is $\mathbb{C}S_k$, where S_k is the symmetric group. If

$$y^{\varepsilon_i^\vee} = $$  then $y^{\varepsilon_i^\vee} y^{\varepsilon_j^\vee} = y^{\varepsilon_j^\vee} y^{\varepsilon_i^\vee}$

and $y^{\varepsilon_1^\vee} \cdots y^{\varepsilon_n^\vee} = T_{w_0}^2 \in Z(H_k(q))$, where $Z(H_k(q))$ is the center of $H_k(q)$.

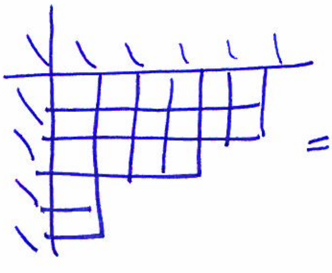
We want to study the tower

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots, \quad \text{where} \quad H_{k-1} \hookrightarrow H_k$$

using

$$\mathrm{Res}^{H_k}_{H_{k-1}} \quad \text{and} \quad \mathrm{Ind}^{H_k}_{H_{k-1}}.$$

A <u>partition</u> is a collection of boxes in a corner

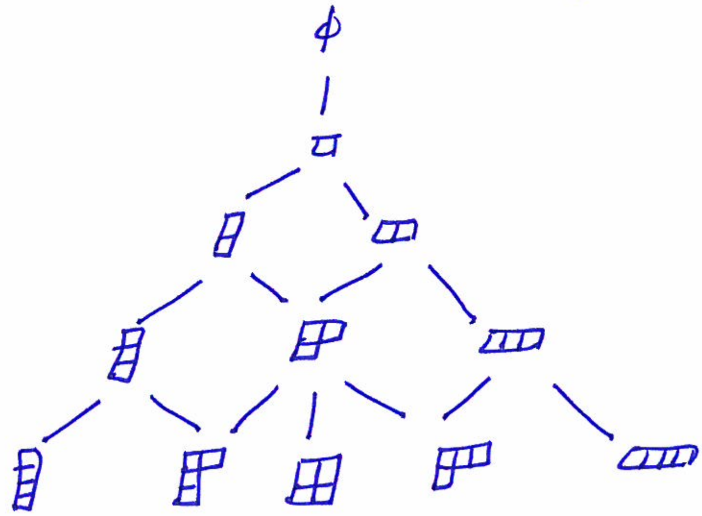
$\lambda = $  $= (6,6,4,1,1).$

Let $\hat{H}_k = \{\text{partitions with } k \text{ boxes}\}$

The <u>Bratelli diagram</u> of the tower $H_1 \subseteq H_2 \subseteq \dots$ has

$\lambda \in \hat{H}_k$ as vertices on level k

$\lambda - \mu$ if μ is obtained by adding a box to λ.



This's means

$$\left\{\begin{matrix}\text{Irreducible} \\ H_k\text{-modules}\end{matrix}\right\} \xleftrightarrow{1-1} \hat{H}_k \quad \text{and}$$

$$\mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\lambda) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu = \square}} H_{k-1}^\mu .$$

Since

$$\mathrm{Hom}_{H_k}(\mathrm{Ind}^{H_k}_{H_{k-1}}(H_{k-1}^\mu), H_k^\lambda) = \mathrm{Hom}_{H_{k-1}}(H_{k-1}^\mu, \mathrm{Res}^{H_k}_{H_{k-1}}(H_k^\lambda))$$

and $\mathrm{Hom}_{H_k}(H_k^\lambda, H_k^\nu) = \begin{cases} 0, & \text{if } \lambda \neq \nu, \\ \mathbb{C}\cdot \mathrm{Id}, & \text{if } \lambda = \nu, \end{cases}$ we get

$$\mathrm{Ind}_{H_{k-1}}^{H_k}\left(H_{k-1}^{\mu}\right) = \bigoplus_{\substack{\lambda \supseteq \mu \\ \lambda/\mu = \square}} H_k^{\lambda}.$$

Note that

$$\dim\left(H_k^{\lambda}\right) = \#\text{ of paths from } \emptyset - \cdots \to \lambda$$

<u>Example</u>: As vector spaces

$$H_4^{(3,1)} = H_3^{(2,1)} \oplus H_3^{(3)} = H_2^{(1,1)} \oplus H_2^{(2)} \oplus H^{(2)}$$

$$= H_1^{\square} \oplus H_1^{\square} \oplus H_1^{\square}.$$

($H_1 \cong M_1(\mathbb{C})$ which has one 1-dimensional simple module)

A <u>standard</u> <u>tableau</u> of shape λ is a filling T of the boxes of λ with $1, 2, \ldots, k$ such that

(a) the rows increase left to right,

(b) the columns increase top to bottom

There is a bijection

$$\left\{\begin{matrix}\text{standard tableaux} \\ \text{of shape } \lambda\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\text{paths } \emptyset - \cdots \to \lambda\right\}$$

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array} \longleftrightarrow \emptyset - \square \to (1,1) \to (2,1) \to (2,2) \to (2,2,1)$$

so that

$$\dim\left(H_k^{\lambda}\right) = \#\text{ of standard tableaux of shape } \lambda.$$

<u>Theorem</u> The irreducible $H_k(q)$-modules are

$$H_k^\lambda = \text{span}\left\{ v_T \;\middle|\; \begin{matrix} T \text{ a standard tableau of} \\ \text{shape } \lambda \end{matrix} \right\}$$

with H_k-action given by

$$y^{\varepsilon_i^\vee} v_T = q^{c(T(i))} v_T,$$

$$T_i v_T = \frac{q-q^{-1}}{1-q^{2(c(T(i))-c(T(i+1)))}} v_T + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(c(T(i))-c(T(i+1)))}}\right) v_{s_iT},$$

where $T(i)$ = box containing i in T,

$c(b) = s-r$, if b is in row r, column s

s_iT is T with i and $i+1$ switched

$v_{s_iT} = 0$ if s_iT is not standard.

<u>Example</u> $H_5^{\boxplus}$ has basis

$$v_{\substack{12\\34\\5}},\quad v_{\substack{12\\35\\4}},\quad v_{\substack{13\\24\\5}},\quad v_{\substack{13\\25\\4}},\quad v_{\substack{14\\25\\3}} \qquad \text{and}$$

$$T_2 v_{\substack{12\\35\\4}} = \frac{q-q^{-1}}{1-q^{2(1-(-1))}} v_{\substack{12\\35\\4}} + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2(1-(-1))}}\right) v_{\substack{13\\25\\4}}$$

$$T_2 v_{\substack{14\\25\\3}} = \frac{q-q^{-1}}{1-q^{2(-1-(-2))}} v_{\substack{14\\25\\3}} + \left(q^{-1} + \frac{q-q^{-1}}{1-q^{2}}\right) v_{\substack{14\\35\\2}}$$

$$= -q^{-1} v_{\substack{14\\25\\3}}.$$

Since

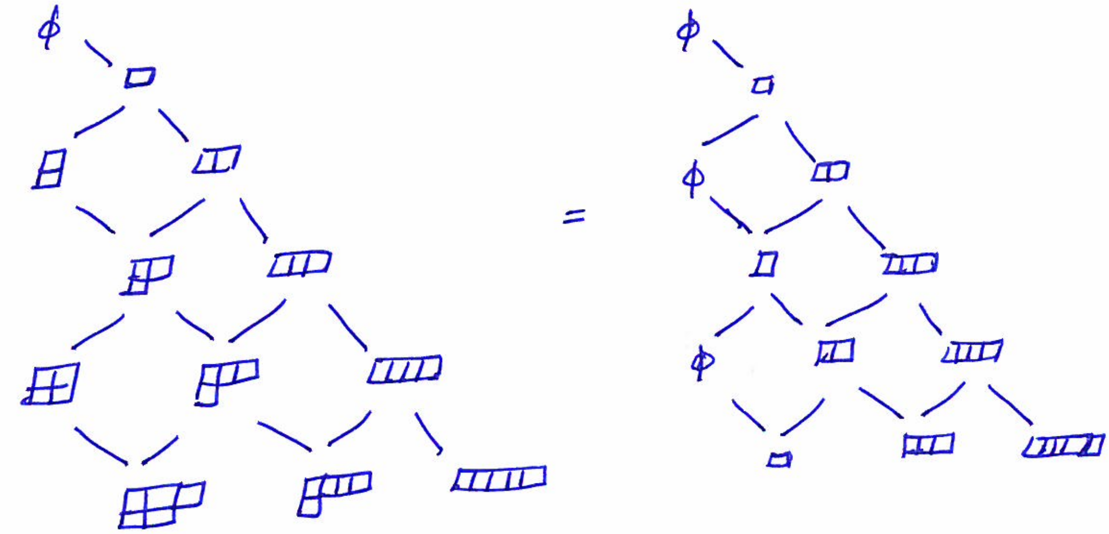
$$\begin{aligned} H_k &\twoheadrightarrow TL_k \\ T_i - q &\longmapsto e_i \end{aligned}$$

is a surjective homomorphism

every TL_k-module is an H_k-module.

The Bratelli diagram for the tower $TL_1 \subseteq TL_2 \subseteq \cdots$

is



<u>Lie algebras</u>

A <u>Lie algebra</u> is a vector space $\mathfrak{g}$ with a bracket $[\,,\,]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

(a) $[x,y] = -[y,x]$, for $x,y \in \mathfrak{g}$

(b) $[x[y,z]] = [[x,y],z] + [y,[x,z]]$,
for $x,y \in \mathfrak{g}$.

A Lie algebra is not an algebra.

The <u>enveloping algebra</u> of $\mathfrak{g}$ is the algebra $U\mathfrak{g}$ generated by the vector space $\mathfrak{g}$ with relations

$$yx = xy - [x,y], \quad \text{for } x,y \in \mathfrak{g}.$$

Example: The Lie algebra sl_2

$$sl_2 = \{ x \in M_2(\mathbb{C}) \mid \mathrm{tr}\, x = 0 \}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a+d=0 \right\}$$

with

$$[x,y] = xy - yx \qquad \left(\begin{matrix}\text{product on the} \\ \text{right is matrix} \\ \text{multiplication}\end{matrix}\right)$$

The vector space sl_2 has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

The enveloping algebra Usl_2 is generated by e, f, h with relations

$$ef = fe + h, \quad eh = he - 2e, \quad hf = fh - 2f$$

The algebra Usl_2 has basis

$$\{ f^{m_1} h^{m_2} e^{m_3} \mid m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0} \}$$

Note: Usl_2 is not far from

$\mathbb{C}[\varepsilon, \varphi, \eta]$, the algebra generated by $\varepsilon, \varphi, \eta$ with relations

$$\varepsilon\varphi = \varphi\varepsilon, \quad \varepsilon\eta = \eta\varepsilon, \quad \eta\varphi = \varphi\eta.$$

<u>$U\mathfrak{sl}_2$ is a Hopf algebra</u>

Let M, N be U-modules.

M has basis $\{m_1, \ldots, m_r\}$

N has basis $\{n_1, \ldots, n_s\}$

The <u>tensor product</u> vector space is

$$M \otimes N \text{ with basis } \left\{ m_i \otimes n_j \;\middle|\; \begin{matrix} 1 \le i \le r \\ 1 \le j \le s \end{matrix} \right\}$$

so that $\dim(M \otimes N) = r \cdot s$. (Note $M \oplus N$ has basis $\{m_1, \ldots, m_r, n_1, \ldots, n_s\}$ and $\dim(M \oplus N) = r + s$.).

U is a Hopf algebra means that it comes with a map

$$\Delta : U \to U \otimes U, \quad \text{the } \underline{\text{coproduct}},$$

that tells me how to make U act on $M \otimes N$.

For $U = U\mathfrak{sl}_2$ this map is

$$\Delta(e) = e \otimes 1 + 1 \otimes e,$$

$$\Delta(f) = f \otimes 1 + 1 \otimes f,$$

$$\Delta(h) = h \otimes 1 + 1 \otimes h.$$

An <u>$\mathfrak{sl}_2$-module</u> is a $U\mathfrak{sl}_2$-module.

Modules for Usl_2

$L(\square)$ has basis $\{v_1, v_{-1}\}$ with

$$\begin{aligned} Usl_2 &\to End(L(\square)) \\ e &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

$L(\square) \otimes L(\square)$ has basis $\{v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1}\}$

and

$$e(v_1 \otimes v_1) = 0 \qquad\qquad e\,v_0 = 2(v_1 \otimes v_1) = 2v_2$$

$$v_0 = f(v_1 \otimes v_1) = v_{-1} \otimes v_1 + v_1 \otimes v_{-1} \qquad e(v_2) = v_0 = v_1 \otimes v_1$$

$$2v_{-2} = f\,v_0 = 2(v_{-1} \otimes v_{-1}) \qquad\qquad e(v_{-1} \otimes v_{-1}) = v_{-1} \otimes v_1 + v_1 \otimes v_{-1}$$

$$\begin{tikzcd} {} \\ v_2 \arrow[u, "e"'] \arrow[d, bend right, "f"'] \\ v_0 \arrow[u, "e"'] \arrow[d, bend right, "f"'] \\ v_{-2} \arrow[u, "e"'] \arrow[d, "f"'] \\ {} \end{tikzcd}$$

$$h\,v_2 = h\,v_1 \otimes v_1 = v_1 \otimes v_1 + v_1 \otimes v_1 = 2v_1 \otimes v_1$$

$$h\,v_0 = 0$$

$$h\,v_{-2} = -2v_{-1} \otimes v_{-1}.$$

and

$$v^0 = v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \quad \text{has} \quad \begin{aligned} e\,v^0 &= 0 \\ f\,v^0 &= 0 \\ h\,v^0 &= 0. \end{aligned}$$

So

$$L(\square) \otimes L(\square) = L(\square\square) \oplus L(\emptyset)$$

where $L(\square\square)$ has basis $\{v_2, v_0, v_{-2}\}$

$L(\emptyset)$ has basis $\{v^0\}$

$L(\phi)\otimes L(\square)$ has basis $\{v^0\otimes v_1,\ v^0\otimes v_{-1}\}$

$$e(v^0\otimes v_1)=0,\qquad e(v^0\otimes v_{-1})=v^0\otimes v_1$$
$$f(v^0\otimes v_1)=v^0\otimes v_{-1},\qquad f(v^0\otimes v_{-1})=\cancel{v^0\otimes}\ 0$$
$$h(v^0\otimes v_1)=v^0\otimes v_1,\qquad h(v^0\otimes v_{-1})=-v^0\otimes v_1$$

So
$$L(\phi)\otimes L(\square)\xrightarrow{\sim}L(\square)$$
$$v^0\otimes v_1\longmapsto v_1$$
$$v^0\otimes v_{-1}\longmapsto v_{-1}.$$

Then $L(\square\square)\otimes L(\square)$ has basis $\left\{\begin{matrix} v_2\otimes v_1, & v_2\otimes v_{-1} \\ v_0\otimes v_1, & v_0\otimes v_{-1} \\ v_{-2}\otimes v_1, & v_{-2}\otimes v_{-1}\end{matrix}\right\}$

and
$$e(v_2\otimes v_1)=0$$
$$v_1=f(v_2\otimes v_1)=v_0\otimes v_1+v_2\otimes v_{-1}$$
$$2v_{-1}=f(v_1)=v_{-2}\otimes v_1+v_0\otimes v_{-1}+v_0\otimes v_{-1}$$
$$3v_{-3}=fv_{-1}=v_{-2}\otimes v_{-1}+v_{-2}\otimes v_{-1}+v_{-2}\otimes v_{-1}.$$

$$\begin{array}{ll}
\uparrow e & \\
v_3=v_2\otimes v_1 & \\
f\downarrow\ \uparrow e & hv_3=3v_3 \\
v_1 & hv_1=v_1 \\
f\downarrow\ \uparrow e & \\
v_{-1} & hv_{-1}=-v_1 \\
f\downarrow\ \uparrow e & \\
v_{-3}=v_{-2}\otimes v_{-1} & hv_{-3}=-3v_3 \\
f\downarrow &
\end{array}$$

and if

$$v'=v_0\otimes v_1-2v_2\otimes v_{-1}\qquad\qquad \begin{array}{c}\uparrow e\\ v'\\ f\downarrow\\ v^{-1}\\ f\downarrow\end{array}$$
$$v^{-1}=2v_{-2}\otimes v_1+v_0\otimes v_{-1}$$
$$-2v_0\otimes v_{-1}=2v_{-2}\otimes v_1-v_0\otimes v_{-1}$$

So $L(\square\square)\otimes L(\square) = L(\square\square\square)\oplus L(\square)$

In general

$$L(\overbrace{\square\cdots\square}^{k})\otimes L(\square) = L(\overbrace{\square\cdots\square}^{k+1})\oplus L(\overbrace{\square\cdots\square}^{k-1})$$

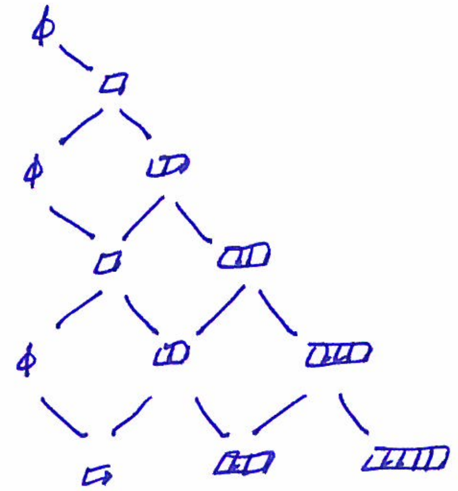
and $U\mathfrak{sl}_2 \longrightarrow \mathrm{End}(L(\overbrace{\square\cdots\square}^{k}))$

$$e \longmapsto \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & & \\ & & 0 & 3 & & \\ & & & 0 & \ddots & \\ & & & & 0 & k \\ & & & & & 0 \end{pmatrix}$$

$$f \longmapsto \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & 3 & \ddots & \\ & & & & 0 \\ & & & k & 0 \end{pmatrix}$$

$$h \longmapsto \begin{pmatrix} k & & & & & & 0 \\ & k-2 & & & & & \\ & & k-4 & & & & \\ & & & \ddots & & & \\ & & & & -(k-4) & & \\ & & & & & -(k-2) & \\ 0 & & & & & & -k \end{pmatrix}$$

The rule for $-\otimes L(\square)$ is given by



So $U\mathfrak{sl}_2$ should have something to do with $TL_1 \subseteq TL_2 \subseteq \cdots$