Week 9 Problem Sheet
Group Theory and Linear algebra
Semester II 2011

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1. Week 9: Vocabulary

(1) Define a dihedral group and give some illustrative examples.

(2) Define a rotation in $\mathbb{R}^2$ and give some illustrative examples.

(3) Define a rotation in $\mathbb{R}^3$ and give some illustrative examples.

(4) Define a G-action on X and give some illustrative examples.

(5) Define a G-set and give some illustrative examples.

(6) Define orbits and stabilizers and give some illustrative examples.

(7) Define the action of G on itself by left multiplication and the action of G on itself by conjugation and give some illustrative examples.

(8) Define conjugate, conjugacy class, and centralizer and give some illustrative examples.

(9) Define the centre of a group and give some illustrative examples.

2. Week 9: Results

(1) Let G be a group and let X be a G-set. Let $x \in X$. Show that the stabilizer of x is a subgroup of G.

(2) Let G be a group and let X be a G-set. Show that the orbits partition G.

(3) Let G be a group and let X be a G-set. Let $x \in X$ and let H be the stabilizer of x. Show that $\mathrm{Card}(G/H) = \mathrm{Card}(Gx)$ and that

$$\mathrm{Card}(G) = \mathrm{Card}(Gx)\mathrm{Card}(H)\,.$$

(4) Let G be a group. Show that G is isomorphic to a subgroup of a permutation group.

(5) Let G be a finite group acting on a finite set X. For each $g \in G$ let $\mathrm{Fix}(g)$ be the set of elements of X fixed by g.

(a) Let $S = \{(g,x) \in G \times X \mid g \cdot x = x\}$. By counting S in two ways, show that
$$|S| = \sum_{x \in X} |\mathrm{Stab}(x)| = \sum_{g \in G} |\mathrm{Fix}(g)|\,.$$
(b) Show that if $g \cdot x = y$ then $g\,\mathrm{Stab}(x)g^{-1} = \mathrm{Stab}(y)$, hence $|\mathrm{Stab}(x)| = |\mathrm{Stab}(y)|$.
(c) Prove that the number of distinct orbits is
$$\frac{1}{|G|}\sum_{g \in G} |\mathrm{Fix}(g)|,$$
i.e. the average number of points fixed by elements of G.

(6) Let G be a finite group. Show that the number of elements of a conjugacy class is equal to the number of cosets of the centralizer of any element of the conjugacy class.

(7) Show that the centre of a group G is a normal subgroup of G.

(8) Let p be a prime, let $n \in \mathbb{Z}_{>0}$ and let G be a group of order p^n. Show that $Z(G) \neq \{1\}$.

(9) Let p be a prime and let G be a group of order p^2. Show that G is isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(10) Let G be a finite group of order divisible by a prime p. Show that G has an element of order p.

(11) Let p be an odd prime and let G be a group of order $2p$. Show that $G \simeq \mathbb{Z}/2p\mathbb{Z}$ or $G \simeq D_p$.

3. Week 9: Examples and computations

(1) Let H denote the subgroup of $D_4 = \langle a, b \mid a^4 = 1, b^2 = 1, bab^{-1} = a^{-1}\rangle$ generated by a. Show that H is a normal subgroup of D_4 and write out the multiplication table of D_4/H.

(2) Let H denote the subgroup of $D_4 = \langle a, b \mid a^4 = 1, b^2 = 1, bab^{-1} = a^{-1}\rangle$ generated by a^2. Show that H is a normal subgroup of D_4 and write out the multiplication table of D_4/H.

(3) Find all of the normal subgroups of D_4.

(4) The **quaternion group** is the set $Q_8 = \{ \pm U, \pm I, \pm J, \pm K\}$ where

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that

$$I^2 = J^2 = K^2 = -U, \quad IJ = K, \quad JK = I, \quad KI = J,$$

and that Q_8 is a subgroup of $\mathrm{GL}_2(\mathbb{C})$.

(5) Find all of the cyclic subgroups of the quaternion group Q_8.

(6) Show that every subgroup of the quaternion group Q_8, except Q_8 itself, is cyclic.

(7) Determine whether Q_8 and D_4 are isomorphic.

(8) Let H denote the subgroup of $D_8 = \langle a, b \rangle$ generated by a^4. Write out the multiplication table of D_8 / H.

(9) Show that the set of rotations in the dihedral group D_n is a subgroup of D_n.

(10) Show that the set of reflections in the dihedral group D_n is not a subgroup of D_n.

(11) Let $n \in \mathbb{Z}_{>0}$. Calculate the order of D_n. Always justify your answers.

(12) Calculate the orders of the elements of D_6. Always justify your answers.

(13) Show that D_3 is isomorphic to S_3.

(14) Show that D_3 is nonabelian and noncyclic.

(15) Prove that D_2 and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are isomorphic.

(16) Let $n \in \mathbb{Z}_{>0}$. Determine the orders of the elements in the dihedral group D_n.

(17) Let $m, n \in \mathbb{Z}_{>0}$ such that $m < n$. Show that D_m is isomorphic to a subgroup of D_n.

(18) Determine if the group of symmetries of a rectangle is a cyclic group.

(19) Show that the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the group D_4 are not isomorphic.

(20) Determine all subgroups of the dihedral group D_5.

(21) Let $n \in \mathbb{Z}_{>0}$. Let $G = D_n$ and $H = C_n$. Compute the cosets of H in G and the index $|G : H|$.

(22) Let D_n be the group of symmetries of a regular n-gon. Let a denote a rotation through $2\pi/n$ and let b denote a reflection. Show that

$$a^n = 1, \qquad b^2 = 1, \qquad bab^{-1} = a^{-1}.$$

Show that every element of D_n has a unique expression of the form a^i or $a^i b$, where $i \in \{0, 1, \ldots, n-1\}$.

(23) Determine all subgroups of the dihedral group D_4 as follows:

(a) Find all the cyclic subgroups of D_4 by considering the subgroup generated by each element.

(b) Find two non-cyclic subgroups of D_4.

(c) Explain why any non-cyclic subgroup of D_4, other than D_4 itself, must be of order 4 and, in fact, must be one of the two subgroups you have listed in the previous part.

(24) Let G be the group of rotational symmetries of a regular tetrahedron so that $|G| = 12$. Show that G has subgroups of order $1, 2, 3, 4$ and 12.

(25) Describe precisely the action of S_n on $\{1, 2, \ldots, n\}$ and the action of $\mathrm{GL}_n(\mathbb{F})$ on $\mathbb{F}^n$.

(26) Describe precisely the action of $\mathrm{GL}_n(\mathbb{F})$ on the set of bases of the vector space $\mathbb{F}^n$ and prove that this action is well defined.

(27) Describe precisely the action of $\mathrm{GL}_n(\mathbb{F})$ on the set of subspaces of the vector space $\mathbb{F}^n$ and prove that this action is well defined.

(28) Find the orbits and stabilisers for the action of S_3 on the set $\{1, 2, 3\}$.

(29) Find the orbits and stabilisers for the action of $G = \mathrm{SO}_2(\mathbb{R})$ on the set $X = \mathbb{R}^2$.

(30) Find the orbits and stabilisers for the action of $G = \mathrm{SO}_3(\mathbb{R})$ on the set $X = \mathbb{R}^3$.

(31) The dihedral group D_6 acts on a regular hexagon. Colour two opposite sides blue and the other four sides red and let G be the subgroup of D_6 which preserves the colours. Let $X = \{A, B, C, D, E, F\}$ be the set of vertices of the hexagon. Determine the stabilizers and orbits for the action of G on X.

(32) Since S_4 acts on $X = \{1, 2, 3, 4\}$ any subgroup G acts on $X = \{1, 2, 3, 4\}$. Let $G = \langle (123) \rangle$. Describe the orbits and stabilizers for the action of G on X.

(33) Since S_4 acts on $X = \{1, 2, 3, 4\}$ any subgroup G acts on $X = \{1, 2, 3, 4\}$. Let $G = \langle (1234) \rangle$. Describe the orbits and stabilizers for the action of G on X.

(34) Since S_4 acts on $X = \{1, 2, 3, 4\}$ any subgroup G acts on $X = \{1, 2, 3, 4\}$. Let $G = \langle (12), (34) \rangle$. Describe the orbits and stabilizers for the action of G on X.

(35) Since S_4 acts on $X = \{1, 2, 3, 4\}$ any subgroup G acts on $X = \{1, 2, 3, 4\}$. Let $G = S_4$. Describe the orbits and stabilizers for the action of G on X.

(36) Since S_4 acts on $X = \{1, 2, 3, 4\}$ any subgroup G acts on $X = \{1, 2, 3, 4\}$. Let $G = \langle(123\ 4), (13)\rangle$ (isomorphic to a dihedral group of order 8). Describe the orbits and stabilizers for the action of G on X.

(37) Let $G = \mathbb{R}$ (with operation addition) and let $X = \mathbb{R}^3$. Let $v \in \mathbb{R}^3$. Show that

$$\alpha \cdot x = x + \alpha v,$$

defines an action of G on X and give a geometric description of the orbits.

(38) Let G be the subgroup of S_{15} generated by the three permutations

$$(1, 12)(3, 10)(5, 13)(11, 15), \quad (2, 7)(4, 14)(6, 10)(9, 13), \quad \text{and} \quad (4, 8)(6, 10)(7, 12)(9, 11).$$

Find the orbits of G acting on $X = \{1, 2, \ldots, 15\}$ and prove that G has order which is a multiple of 60.

(39) Let G be a group of order 5 acting on a set X with 11 elements. Determine whether the action of G on X has a fixed point.

(40) Let G be a group of order 15 acting on a set X with 8 elements. Determine whether the action of G on X has a fixed point.

(41) Give an explicit isomorphism between D_2 and a subgroup of S_4.

(42) Find the conjugacy classes of D_4.

(43) Find the centre of D_4.

(44) Let G be a group. Show that $\{1\} \subseteq Z(G)$.

(45) Show that $Z(S_3) = \{1\}$.

(46) Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Determine the centre of $GL_n(\mathbb{F})$.

(47) Find the conjugacy classes in the quaternion group.

(48) Find the conjugates of (123) in S_3 and find the conjugates of (123) in S_4.

(49) Find the conjugates of (1234) in S_4 and find the conjugates of (1234) in S_n, for $n \geq 4$.

(50) Find the conjugates of $(12\ldots m)$ in S_n, for $n \geq m$.

(51) Describe the conjugacy classes in the symmetric group S_n.

(52) Suppose that g and h are conjugate elements of a group G. Show that $C_G(g)$ and $C_G(h)$ are conugate subgroups of G.

(53) Determine the centralizer in $\mathrm{GL}_3(\mathbb{R})$ of the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(54) Determine the centralizer in $\mathrm{GL}_3(\mathbb{R})$ of the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(55) Determine the centralizer in $\mathrm{GL}_3(\mathbb{R})$ of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(56) Determine the centralizer in $\mathrm{GL}_3(\mathbb{R})$ of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(57) Let G be a group and assume that $G/Z(G)$ is a cyclic group. Show that G is abelian.

(58) Describe the finite groups with exactly one conjugacy class.

(59) Describe the finite groups with exactly two conjugacy classes.

(60) Describe the finite groups with exactly three conjugacy classes.

(61) Let p be a prime. Show that a group of order p^2 is abelian.

(62) Let p be a prime and let G be a group of order p^2. Show that $G \simeq Z/p\,\mathbb{Z} \times Z/p\,\mathbb{Z}$ or $G \simeq Z/p^2\,\mathbb{Z}$.

(63) Let p be a prime and let G be a group of order $2p$. Show that G has a subgroup of order p and that this subgroup is a normal subgroup.

(64) Let p be a prime. Show that, up to isomorphism, there are exactly two groups of order 2

p.

(65) Prove that every nonabelian group of order 8 is isomorphic to the dihedral group D_4 or to the quaternion group Q_8.

(66) Show that each group G acts on $X = G$ by right multiplication: $g \cdot x = xg^{-1}$, for $g \in G$, $x \in X$.

(67) Let $G = D_2$ act as symmetries of a rectangle. Determine the stabilizer and orbit of a vertex, and the stabilizer and orbit of the midpoint of an edge.

(68) Let $\mathrm{GL}_2(\mathbb{R})$ act on $\mathbb{R}^2$ in the usual way: $A \cdot \overrightarrow{x} = A\overrightarrow{x}$, for $A \in \mathrm{GL}_2(\mathbb{R})$ and $\overrightarrow{x}$ a column vector in $\mathbb{R}^2$. Determine the stabilizer and orbit of $(0, 0)$ and the stabilizer and orbit of $(1, 0)$.

(69) Let G be the group of rotational symmetries of a regular tetrahedron T.

(a) For the action of G on T, describe the stabilizer and orbit of a vertex, and describe the stabilizer and orbit of the midpoint of an edge.
(b) Use the results of (a) to calculate the order of G in two different ways.
(c) By considering the action of G on the set of vertices of T, find a subgroup of S_4 isomorphic to G.

(70) A group G of order 9 acts on a set X with 16 elements. Show that there must be at least one point in X fixed by all elements of G (i.e. an orbit consisting of a single element).

(71) Find the conjugacy class and centralizer of (12) and (123) in S_3. Check that |conjugacy class| $\cdot$ |centralizer| = $|S_3|$ in each case.

(72) Let τ be a permutation in S_m.

(a) Let σ be an n-cycle $\sigma = (a_1 a_2 \cdots a_n)$ in S_m. Show that $\tau\sigma\tau^{-1}$ takes $\tau(a_1) \mapsto \tau(a_2)$, $\tau(a_2) \mapsto \tau(a_3)$, ..., $\tau(a_n) \mapsto \tau(a_1)$. Hence $\tau\sigma\tau^{-1}$ is the n-cycle $(\tau(a_1)\tau(a_2)\cdots\tau(a_n))$.
(b) Use the previous result to find all conjugates of (123) in S_4.
(c) Find a permutation τ in S4 conjugating $\sigma = (1234)$ to $\tau\sigma\tau^{-1} = (2413)$.
(d) If $\sigma = \sigma_1 \cdots \sigma_k$, show that $\tau\sigma\tau^{-1} = \tau\sigma_1\tau^{-1} \cdots \tau\sigma_k\tau^{-1}$.
(e) Use the previous results to find all conjugates of (12)(34) in S_4.

(73) Find the number of conjugacy classes in each of S_3, S_4 and S_5 and write down a representative from each conjugacy class. How many elements are in each conjugacy class?

(74) Let H be a subgroup of G. Show that H is a normal subgroup of G if and only if H is a union of conjugacy classes in G.

(75) Find normal subgroups of S_4 of order 4 and of order 12.

(76) Find the centralizer in $GL_2(\mathbb{R})$ of the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

(77) Show that $SL_2(\mathbb{R})$ acts on the upper half plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Prove that this action is well defined and describe the orbit and stabiliser of i.

4. References

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[Ra] A. Ram, *Notes in abstract algebra*, University of Wisconsin, Madison 1994.