Week 12 Problem Sheet
Group Theory and Linear algebra
Semester II 2011

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010 Australia
aram@unimelb.edu.au

Last updates: 1 September 2011

1. Week 12: Questions from past exams

(1) Consider the permutation group $G = \{(1),(12)(34),(13)(24),(14)(23)\}$ acting on a set X of four symbols 1,2,3,4.

(a) Describe the orbit and stabiliser of 1. Explain how the orbit/stabiliser theorem connects G and the orit and stabiliser.

(b) Find the orbit and stabiliser of 1 for the action of the subgroup $H = \{(1),(12)(34)\}$ acting on the set X.

(2)

(a) If a group of order 9 acts on a set X with 4 elements, explain why each orbit must consist of either one or three points.

(b) Explain why a group with 9 elements must have an element in the centre, which is different from the identity element.

(3) Let V be a complex finite dimensional inner product space and let $f: V \to V$ be a linear transformation satisfying $f^* f = f f^*$.

(a) State the spectral theorem and deduce that there is an orthonormal basis of V consisting of eigenvectors of f.

(b) Show that there is a linear transformation $g: V \to V$ so that $f = g^2$.

(c) Show that if every eigenvalue of f has absolute value 1, then $f^* = f^{-1}$.

(d) Give an example to show that the result in (a) can fail if V is a real inner product space.

(4)

(a) Let A be an $n \times n$ complex Hermitian matrix. Define a product on $\mathbb{C}^n$ by $(X, Y) = XAY^*$, where $X, Y \in \mathbb{C}^n$ are written as row vectors. Show that this is an inner product if all the eigenvalues of A are positive real numbers.

(b) Show that if $A = B^*B$, where B is any invertible $n \times n$ complex matrix, then A is a Hermitian matrix and all the eigenvalues of A are real and positive.

(5) Let G be the multiplicative group $\mathrm{GL}_2(\mathbb{F}_2)$ of invertible 2×2 matrices, where the entries are from the field with two elements $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. There are six matrices which are elements of this group.

Let V be the 2-dimensional vector space over the field $\mathbb{F}_2$ (V contains 4 vectors). Then G acts on V by usual multiplication of column vectors by matrices; $A: X \to AX$, where $A \in G, X \in V$.

(a) Find the orbits and stabilisers of the vectors $(0,0)^t$ and $(1,0)^t$ under the action of G, where the transpose t converts row vectors to column vectors.

(b) Use this action to construct a homomorphism φ from G into S_4, the permutation group on 4 symbols.

(c) Prove that the homomorphism φ is injective.

(6) Consider the symmetric group S_4 acting on the four numbers {1,2,3,4}. Consider the three ways of dividing these numbers into two pairs, namely $P_1 = \{\{1,2\},\{3,4\}\}$, $P_2 = \{\{1,3\},\{2,4\}\}$, $P_3 = \{\{1,4\},\{2,3\}\}$.

(a) Construct a homomorphism from S_4 onto S_3 by using the action of S_4 on {1,2,3,4} to give an action of S_4 on the set of three objects $\{P_1, P_2, P_3\}$. In particular, explain why the mapping you have described is a homomorphism.

(b) Describe the elements of the kernel K of this homomorphism and explain why this subgroup is normal.

(c) Explain why the quotient group S_4/K is isomorphic to S_3.

(7) Consider the infinte pattern of symbols

...YYYYYYYYYYY...

(a) Describe the full group G of symmetries of this pattern.

(b) Describe the stabiliser H of one of the symbols Y.

(c) Describe the maximal normal subgroup of translations T in G and explain why the quotient group G/T is isomorphic to the stabiliser subgroup H.

(8) An inner product $\langle , \rangle$ on $\mathbb{R}^3$ is defined by

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 .$$

Let W be the subspace of $\mathbb{R}^3$ spanned by $\{(1, -1, 0), (0, 1, -1)\}$. Find all vectors in W orthogonal to $(1, 1, -1)$.

(9) The subset $\{1, 2, 4, 5, 7, 8\}$ of $\mathbb{Z}/9\mathbb{Z}$ forms a group G under multiplication modulo 9.

(a) Show that the group G is cyclic.
(b) Give an example of a non-cyclic group of order 6.

(10)

(a) Express the following permutations as products of disjoint cycles: (134)(25) $\cdot$ (12345) and the inverse of (12)(3456).
(b) Find the order of the permutation (123)(4567).

(11) Let G be a group of order 21.

(a) What are the possible orders of subgroups of G?
(b) What are the possible orders of non-cyclic subgroups of G?

Always explain your answers.

(12)

(a) Show that the set

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \; a, b \in \mathbb{R}, a \neq 0 \right\}$$

forms a group G under matrix multiplication.
(b) Show that the function $f : G \to \mathbb{R}^*$ defined by

$$f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = a^2$$

is a homomorphism from G to the multiplicative group $\mathbb{R}^*$ of non-zero real numbers.
(c) Find the image and kernel of f.

(13) A group G of order 8 acts on a set X consisting of 11 points.

(a) What are the possible sizes of orbits?
(b) Show that there must be a point of X fixed by all elements of G.

Always explain your answers.

(14) Let V be a complex inner product space and let $T: V \to V$ be a linear transformation such that $T^*T = TT^*$.

(a) Explain how the adjoint T^* of T is defined.
(b) Prove that $\|Tx\| = \|T^*x\|$ for all $x \in V$.
(c) Deduce that the nullspace of T^* is equal to the nullspace of T.

(15) Let A be an $n \times n$ complex matrix.

(a) Prove that if A is Hermitian, then all eigenvalues of A are real.
(b) Carefully state the spectral theorem for normal matrices. Use this to show that if A is a normal matrix with all real eigenvalues, then A is Hermitian.

(16) Let X be the graph of $y = \sin x$ in the x-y plane,

$$X = \{(x, y) \in \mathbb{R}^3 \mid y = \sin x\},$$

and let G be the symmetry group of X.

(a) Describe all the symmetries in G.
(b) Find the orbit and stabilizer of the point $(0, 0)$ under the action of G on X.
(c) Find the translational subgroup T of G.
(d) Explain why T is a normal subgroup of G.

(17) Let G be the subgroup of the symmetric group S_4 consisting of the permutations

$$\{ \quad (1), (12)(34), (13)(24), (14)(23)$$
$$(123), (132), (124), (142), (134), (143), (234), (243) \quad \}$$

(a) Show that G has 4 conjugacy classes, containing $1, 3, 4$ and 4 elements.
(b) Explain why any normal subgroup of G is a union of conjugacy classes.
(c) Deduce that G contains no normal subgroup of order 6.
(d) Does G contain any subgroup of order 6?

Always explain your answers.

(18)

(a) Show that if G is a group with centre Z such that G/Z is cyclic, then G is abelian.

(b) If G is a nonabelian group of order p^3 where p is prime, what can you say about the centre Z of G and the quotient group G/Z?

Always explain your answers.

(19) Let $V = \mathscr{P}_2(\mathbb{R})$ be the real vector space of all polynomials of degree ≤ 2 with real coefficients. An inner product $\langle , \rangle$ on V is defined by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx\,.$$

Find a basis for the orthogonal complement of the subspace W spanned by $\{1, x\}$.

(20) Consider the complex matrix

$$A = \begin{pmatrix} 1 & 1 \\ i & 1 \end{pmatrix}.$$

Decide whether the matrix is: (i) Hermitian, (ii) unitary, (iii) normal, (iv) diagonalizable. Always explain your answers.

(21) The set of eight elements $\{\pm 1, \pm 2, \pm 4, \pm 7\}$ forms a group G under multiplication modulo 15.

(a) Find the order of each element in G.

(b) Is the group cyclic?

Always explain your answers.

(22)

(a) Express the permutation $(1342) \cdot (345)(12)$ as a product of disjoint cycles.

(b) Find the order of the permutation $(12)(4536)$ in the group S_6.

(c) Find all the conjugates of (123) in the group S_3.

(23) Let G be a finite group containing a subgroup H of order 4 and a subgroup K of order 7.

(a) State Lagrange's theorem for finite groups.

(b) What can you say about the order of G?

(c) What can you say about the order of the subgroup $H \cap K$?

Always explain your answers.

(24)

(a) Show that

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$$

is a subgroup of $\mathrm{GL}_3(\mathbb{R})$ using matrix multiplication as the operation.

(b) Find the centre of G.

(25) Let G be the group of symmetries of the rectangle X with vertices $(2,1), (2,-1), (-2,1)$ $, (-2,-1)$.

(a) Give geometric descriptions of the symmetries in G.

(b) Find the orbit and stabilizer of the point $Q = (2,0)$ under the action of G on X.

(c) Check that your answers to parts (a) and (b) are consistent with the orbit-stabiliser theorem.

(26) Let $f : V \to V$ be a linear operator on a finite dimensional inner product space.

(a) Explain how the adjoint f^* of f is defined.

(b) Prove that the nullspace of f^* is the orthogonal complement of the range of f .

(c) Deduce that the nullity of f^* is equal to the nullity of f.

(27) Consider the complex matrix

$$A = \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix}.$$

(a) Without calculating eigenvalues, explain why A is diagonalizable.

(b) Find a diagonal matrix D and a unitary matrix U such that

$$U^{-1}AU = D.$$

(c) Write down U^{-1}.

(d) Find a complex matrix B such that $B^2 = A$.

(28) Let G be a group in which every element has order 1 or 2.

(a) Prove that G is abelian.

(b) Prove that if G is finite then G has order 2^n for some integer $n \in \mathbb{Z}_{\geq 0}$.

(c) For each integer $n \geq 1$, give an example of a group of order 2^n with each element of order 1 or 2.

Always explain your answers.

(29) For any isometry $f : \mathbb{E}^2 \to \mathbb{E}^2$ of the Euclidean plane, let

$$\mathrm{Fix}(f) = \{x \in \mathbb{E}^2 \mid f(x) = x\}$$

denote the fixed point set of f.

(a) Show that if f and g are isometries of $\mathbb{E}^2$ then

$$\mathrm{Fix}(gfg^{-1}) = g\,\mathrm{Fix}(f).$$

(b) The non-identity isometries of $\mathbb{E}^2$ are of four types: rotations, reflections, translations, and glide reflections. Describe the fixed point set for each type.

(c) Deduce from parts (a) and (b) that if f is a rotation about a point p then gf g^{-1} is a rotation about the point $g(p)$.

(30) Let $\mathbb{Q}$ denote the additive group of rational numbers, and $\mathbb{Z}$ the subgroup of integers.

(a) Show that every element of the quotient group $\mathbb{Q}/\mathbb{Z}$ has finite order.

(b) Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the multiplicative group of complex numbers of absolute value one. Show that the function $f : \mathbb{Q} \to S^1$ defined by

$$f(x) = e^{2\pi i x} = \cos(2\pi x) + i \sin(2\pi x)$$

is a homomorphism.

(c) Find the kernel of f.

(d) Deduce that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to a subgroup of S^1.

(e) Is $\mathbb{Q}/\mathbb{Z}$ is isomorphic to S^1?

Always explain your answers.

(31)

(a) Use the Euclidean algorithm to find $d = \gcd(469, 959)$.

(b) Find integers x, y such that $469x + 959y = d$.

(32) The complex vector space $\mathbb{C}^4$ has an inner product defined by

$$\langle a, b \rangle = a_1\overline{b}_1 + a_2\overline{b}_2 + a_3\overline{b}_3 + a_4\overline{b}_4$$

for $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4) \in \mathbb{C}^4$. Let W be the subspace of $\mathbb{C}^4$ spanned by the vectors $(1, 0, -1, 0)$ and $(0, 1, 0, i)$.

Find a basis for the orthogonal complement $W^{\perp}$ of W.

(33) Determine whether the matrix $A = \begin{pmatrix} 3 & 4i \\ 4i & 3 \end{pmatrix}$ is (i) Hermitian, (ii) unitary, (iii) normal, (iv) diagonalizable. Always explain your answers.

(34) The sets $G_1 = \{1, 3, 9, 11\}$ and $G_2 = \{1, 7, 9, 15\}$ form groups under multiplication modulo 16.

(a) Find the order of each element in G_1 and each element in G_2.

(b) Are the groups G_1 and G_2 isomorphic?

Always explain your answers.

(35)

(a) Express the following permutation as a product of disjoint cycles: (234)(56)* (1354)(26).

(b) Find the order of the permutation (12)(34567) in S_7.

(c) Find all conjugates of (13)(24) in the group S_4.

(36) Let G be a group of order 35.

(a) What does Lagrange's theorem tell you about the orders of subgroups of G?

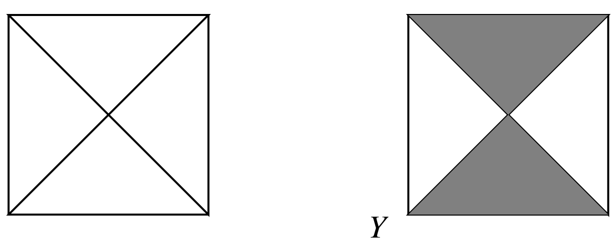
(b) If H is a subgroup of G with $H \neq G$, expalin why H is cyclic.

(37) Consider the set of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| \; a, b \in \mathbb{R}, a^2 - b^2 = 1 \right\}.$$

Prove that G is a group using matrix multiplication as the operation.

(38) Let X be a subset of $\mathbb{R}^2$ consisting of the four edges of a square together with its two diagonals. Let Y be obtained from X by filling in two triangles as shown below:



X Y

Let G be the symmetry group of X and H the symmetry group of Y.

(a) Describe the group G by giving geometric descriptions of the symmetries in G, and writing down a familiar group isomorphic to G.

(b) Give a similar description of H.

(c) Explain why H is a normal subgroup of G.

(39) Let $f: V \to V$ be a self-adjoint linear operator on an inner product space V, i.e. $f^* = f$.

(a) Prove that every eigenvalue of f is real.

(b) Let v_1, v_2 be eigenvectors of f corresponding to eigenvalues λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. Prove that v_1 and v_2 are orthogonal.

(40) Let A be a 6×6 complex matrix with minimal polynomial

$$m(X) = (X+1)^2(X-1)\,.$$

(a) Describe the possible characteristic polynomials for A.

(b) Let the possible Jordan normal forms for A (up to reordering the Jordan blocks).

(c) Explain why A is invertible and write A^{-1} as a polynomial in A.

(41)

(a) Let $f: V \to V$ be a normal linear operator on a complex inner product space V such that $f^4 = f^3$. Use the spectral theorem to prove that f is self-adjoint and that $f^2 = f$.

(b) Give an example of a linear operator $g: V \to V$ on a complex inner product space V such that $g^4 = g^3$ but $g^2 \neq g$.

(42) Consider the subgroup $H = \{\pm 1, \pm i\}$ of the multiplicative group $G = \mathbb{C}^*$ of non-zero complex numbers.

(a) Describe the cosets of H in G. Draw a diagram in the complex plane showing a typical coset.

(b) Show that the function $f: G \to G$ defined by $f(z) = z^4$ is a homomorphism and find its kernel and image.

(c) Explain why H is a normal subgroup of G and identify the quotient group G/H.

(43) Let G be the cyclic subgroup of S_7 generated by the permutation (12)(3456). Consider the action of G on $X = \{1, 2, 3, 4, 5, 6, 7\}$.

(a) Write down all the elements of G.

(b) Find the orbit and stabilizer of (i) 1, (ii) 3 and (iii) 7. Check that your answers are consistent with the orbit-stabilizer theorem.

(c) Prove that if a group H of order 4 acts on a set Y with 7 elements then there must be at least one element of Y fixed by all elements of H.

(44) Let p be a prime number, and let V be the vector space over the field $\mathbb{Z}/p\mathbb{Z}$ consisting of all column vectors in $(\mathbb{Z}/p\mathbb{Z})^2$:

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x, y \in \mathbb{Z}/p\mathbb{Z} \right\}.$$

Let $G = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices with $\mathbb{Z}/p\mathbb{Z}$ entries using matrix multiplication.This acts on V by matrix multiplication as usual: $A \cdot v = Av$ for all $A \in G$ and all $v \in V$.

(a) Consider the 1-dimensional subspaces of V. Show that there are exactly $p+1$ such subspaces: spanned by the vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ldots \begin{pmatrix} p-1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Explain why G also acts on the set X of 1-dimensional subspaces of V. This gives a homomorphism $\varphi : G \to S_{p+1}$.

(c) Show that the kernel of φ consists of the scalar matrices

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \; a \in \mathbb{Z}/p\mathbb{Z} - \{0\} \right\}.$$

Deduce that the quotient group G/K is isomorphic to a subgroup of S_{p+1}.

(d) For the case where $p = 3$, find $|K|$ and $|G|$. Deduce that G/K is isomorphic to S_4.

2. References

[GH] J.R.J. Groves and C.D. Hodgson, *Notes for 620-297: Group Theory and Linear Algebra*, 2009.

[Ra] A. Ram, *Notes in abstract algebra*, University of Wisconsin, Madison 1994.