The Basic Trigonometric Identities

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010 Australia
aram@unimelb.edu.au
and

Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Last updates: 2 March 2010

1. The Basic Trigonometric Identities

The **exponential** expression is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots.$$

The **sine** and **cosine** expressions are

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \text{and} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{with} \qquad i^2 = -1.$$

The **tangent**, **cotangent**, **secant** and **cosecant** expressions are

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{1}{\tan x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}.$$

Example. Prove that $e^{ix} = \cos x + i \sin x$.

⊟ *Proof*.

$$\begin{aligned} \cos x + i \sin x &= \frac{e^{ix} + e^{-ix}}{2} + i\left(\frac{e^{ix} - e^{-ix}}{2i}\right) \\ &= \frac{e^{ix} + e^{-ix} + e^{ix} - e^{-ix}}{2} \\ &= e^{ix}. \end{aligned}$$

□

Example. Prove that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots \qquad \text{and}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots.$$

⊟ *Proof*.

Compare coefficients of 1 and i on each side of

$$\begin{aligned}
\cos x + i\sin x = \quad & e^{ix} \\
= \quad & 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\
= \quad & 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots \\
= \quad & 1 + ix + \frac{i^2x^2}{2!} + \frac{i\cdot i^2x^3}{3!} + \frac{\left(i^2\right)^2x^4}{4!} + \frac{i\cdot\left(i^2\right)^2x^5}{5!} + \frac{\left(i^2\right)^3x^6}{6!} + \frac{i\cdot\left(i^2\right)^3x^7}{7!} + \cdots \\
= \quad & 1 + ix + \frac{(-1)x^2}{2!} + \frac{i\cdot(-1)x^3}{3!} + \frac{(-1)^2x^4}{4!} + \frac{i\cdot(-1)^2x^5}{5!} + \frac{(-1)^3x^6}{6!} + \frac{i\cdot(-1)^3x^7}{7!} + \cdots \\
= \quad & 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + i\frac{x^7}{7!} + \cdots \\
= \quad & \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right).
\end{aligned}$$

□

Example. Prove that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.

⊞ *Proof*.

$$\begin{aligned}
\cos(-x) &= \frac{e^{i(-x)} + e^{-i(-x)}}{2} \\
&= \frac{e^{-ix} + e^{ix}}{2} \\
&= \cos x.
\end{aligned}$$

and

$$\begin{aligned}
\sin(-x) &= \frac{e^{i(-x)} - e^{-i(-x)}}{2i} \\
&= \frac{e^{-ix} - e^{ix}}{2i} \\
&= -\sin x.
\end{aligned}$$

□

Example. Prove that $\cos^2 x + \sin^2 x = 1$.

⊟ *Proof*.

$$
\begin{aligned}
1 &= e^0 \\
&= e^{ix+(-ix)} \\
&= e^{ix}e^{-ix} \\
&= e^{ix}e^{i(-x)} \\
&= (\cos x + i\sin x)(\cos(-x) + i\sin(-x)) \\
&= (\cos x + i\sin x)(\cos x - i\sin x) \\
&= \cos^2 x - i\sin x\cos x + i\sin x\cos x - i^2\sin^2 x \\
&= \cos^2 x - (-1)\sin^2 x \\
&= \cos^2 x + \sin^2 x.
\end{aligned}
$$

□

Example. Prove that

$$
\begin{aligned}
\cos(x+y) &= \cos x\cos y - \sin x\sin y, \qquad \text{and} \\
\sin(x+y) &= \sin x\cos y + \cos x\sin y.
\end{aligned}
$$

⊟ *Proof*.

Comparing the coefficients of 1 and i on each side of

$$
\begin{aligned}
\cos(x+y) + i\sin(x+y) &= e^{i(x+y)} \\
&= e^{ix+iy} \\
&= e^{ix}e^{iy} \\
&= (\cos x + i\sin x)(\cos y + i\sin y) \\
&= \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y \\
&= \cos x\cos y + i\cos x\sin y + i\sin x\cos y - \sin x\sin y \\
&= (\cos x\cos y - \sin x\sin y) + i(\cos x\sin y + \sin x\cos y).
\end{aligned}
$$

gives

$$
\begin{aligned}
\cos(x+y) &= \cos x\cos y - \sin x\sin y, \qquad \text{and} \\
\sin(x+y) &= \sin x\cos y + \cos x\sin y.
\end{aligned}
$$

□

The **hyperbolic sine** and **hyperbolic cosine** expressions are

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

respectively. We also define the **hyperbolic tangent**, **hyperbolic cotangent**, **hyperbolic secant** and **hyperbolic cosecant** functions by

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \text{and} \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$

respectively.

Example. Prove that $e^x = \cosh x + \sinh x$.

⊟ *Proof.*

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) \\
&= \cosh x + \sinh x.
\end{aligned}
$$

□

Example. Prove that $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$.

⊟ *Proof.*

$$
\begin{aligned}
\cosh(-x) &= 1 + \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} + \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} + \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} + \cdots \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots \\
&= \cosh x
\end{aligned}
$$

and

$$
\begin{aligned}
\sinh(-x) &= (-x) + \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} + \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} + \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} + \cdots \\
&= -x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} - \frac{x^{11}}{11!} - \frac{x^{13}}{13!} - \cdots \\
&= -\sinh x.
\end{aligned}
$$

□

Example. Prove that

$$
\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots
$$

$$
\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots.
$$

⊟ *Proof.*

$$\begin{aligned}
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
&= \frac{1}{2}\Bigg(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad +1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots\Bigg) \\
&= \frac{1}{2}\Bigg(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad +1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\Bigg) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \\
\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\
&= \frac{1}{2}\Bigg(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad -1 - (-x) - \frac{(-x)^2}{2!} - \frac{(-x)^3}{3!} - \frac{(-x)^4}{4!} - \frac{(-x)^5}{5!} - \frac{(-x)^6}{6!} - \frac{(-x)^7}{7!} - \cdots\Bigg) \\
&= \frac{1}{2}\Bigg(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad -1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \cdots\Bigg) \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots
\end{aligned}$$

□

Example. Prove that $\cosh^2 x - \sinh^2 x = 1$.

⊟ *Proof*.

$$\begin{aligned}
1 &= e^0 \\
&= e^{x+(-x)} \\
&= e^x e^{-x} \\
&= (\cosh x + \sinh x)(\cosh(-x) + \sinh(-x)) \\
&= (\cosh x + \sinh x)(\cosh x - \sinh x) \\
&= \cosh^2 x - \sinh x \cosh x + \sinh x \cosh x - \sinh^2 x \\
&= \cosh^2 x - \sinh^2 x.
\end{aligned}$$

Example. Prove that

$$\begin{aligned}
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \qquad \text{and} \\
\sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y.
\end{aligned}$$

⊟ *Proof*.

We have

$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y + e^{-x} e^y + e^x e^{-y} + e^{-x} e^{-y}}{4} + \frac{e^x e^y - e^{-x} e^y - e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&= \frac{2e^x e^y + 2e^{-x} e^{-y}}{4} \\
&= \frac{e^x e^y + e^{-x} e^{-y}}{2} \\
&= \frac{e^{x+y} + e^{-(x+y)}}{2} \\
&= \cosh(x+y)
\end{aligned}$$

and

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y - e^{-x} e^y + e^x e^{-y} - e^{-x} e^{-y}}{4} + \frac{e^x e^y + e^{-x} e^y - e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&= \frac{2e^x e^y - 2e^{-x} e^{-y}}{4} \\
&= \frac{e^x e^y - e^{-x} e^{-y}}{2} \\
&= \frac{e^{x+y} - e^{-(x+y)}}{2} \\
&= \sinh(x+y).
\end{aligned}$$