Polynomials

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1. Polynomials

Let $\mathbb{F}$ be a field. If $a_0, a_1, a_2, \ldots \in \mathbb{F}$ use the notation

$$a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i.$$

The **polynomial ring** is the set

$$\mathbb{F}[x] = \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \;\middle|\; a_i \in \mathbb{F} \text{ and all but a finite number of the } a_i \text{ are equal to } 0 \right\}$$

with operations given by

$$\left(\sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \right) + \left(\sum_{i \in \mathbb{Z}_{\geq 0}} b_i x^i \right) = \left(\sum_{i \in \mathbb{Z}_{\geq 0}} (a_i + b_i) x^i \right)$$

and

$$\left(\sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \right) \left(\sum_{j \in \mathbb{Z}_{\geq 0}} b_j x^j \right) = \left(\sum_{k \in \mathbb{Z}_{\geq 0}} c_k x^k \right), \qquad \text{where} \qquad c_k = \sum_{i+j=k} a_i b_j.$$

The **degree** function is $\deg : \mathbb{F}[x] \to \mathbb{Z}_{\geq 0}$, where

$$\deg\left(p_0 + p_1 x + p_2 x^2 + \ldots\right) \text{ is the maximal nonnegative integer } d \text{ such that } p_d \neq 0.$$

Let $a \in \mathbb{F}$. The **evaluation homomorphism** is

$$\begin{array}{rcl} \mathrm{ev}_a : \mathbb{F}[x] & \longrightarrow & \mathbb{F} \\ p(x) & \longmapsto & p(a), \end{array}$$

where

$$p(a) = p_0 + p_1 a + p_2 a^2 + \ldots \qquad \text{if} \qquad p(x) = p_0 + p_1 x + p_2 x^2 + \ldots.$$

The **ring of formal power series** in x is

$$\mathbb{F}[[x]] = \left\{ \sum_{i \in \mathbb{Z}_{\geq 0}} a_i x^i \;\middle|\; a_i \in \mathbb{F} \right\},$$

with operations given by

$$\left(\sum_{i\in\mathbb{Z}_{\geq 0}} a_i x^i\right)+\left(\sum_{i\in\mathbb{Z}_{\geq 0}} b_i x^i\right)=\left(\sum_{i\in\mathbb{Z}_{\geq 0}} (a_i+b_i)x^i\right)$$

and

$$\left(\sum_{i\in\mathbb{Z}_{\geq 0}} a_i x^i\right)\left(\sum_{j\in\mathbb{Z}_{\geq 0}} b_j x^j\right)=\left(\sum_{k\in\mathbb{Z}_{\geq 0}} c_k x^k\right), \qquad \text{where} \qquad c_k=\sum_{i+j=k} a_i b_j.$$

Examples. The following are elements of $\mathbb{F}[[x]]$:

(1) $$\frac{1}{1-x}=1+x+x^2+x^3+\cdots,$$

(2) $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\sum_{i\in\mathbb{Z}_{\geq 0}}\frac{x^{\text{i}}}{i!},$$

(3) $$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots=\sum_{i\in\mathbb{Z}_{\geq 0}}(\text{-}1)^i\frac{x^{(2i+1)}}{(2i+1)!},$$

(4) $$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\sum_{i\in\mathbb{Z}_{\geq 0}}(\text{-}1)^i\frac{x^{2i}}{(2i)!},$$

(5) $$\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots=\sum_{i\in\mathbb{Z}_{>0}}\frac{x^i}{i}.$$

Proposition 1.1

a. $\mathbb{F}[x]$ *is an integral domain*.
b. $\mathbb{F}[[x]]$ *is an integral domain*.

HW: Show that $\mathbb{F}[x]$ with the degree function deg is a Euclidean domain.

The **field of fractions of** $\mathbb{F}[x]$ is the set

$$\mathbb{F}(x)=\left\{\frac{a(x)}{b(x)}\;\middle|\; a(x),\, b(x)\in\mathbb{F}[x],\, b(x)\neq 0\right\},$$

with

$$\frac{a(x)}{b(x)}=\frac{c(x)}{d(x)}, \qquad \text{if} \qquad a(x)d(x)=b(x)c(x),$$

and with operations given by

$$\frac{a(x)}{b(x)}+\frac{c(x)}{d(x)}=\frac{a(x)d(x)+b(x)c(x)}{b(x)d(x)} \qquad \text{and} \qquad \frac{a(x)}{b(x)}\cdot\frac{c(x)}{d(x)}=\frac{a(x)c(x)}{b(x)d(x)}.$$

The **field of fractions of** $\mathbb{F}[[x]]$ is the set

$$\mathbb{F}((x))=\left\{\frac{a(x)}{b(x)}\;\middle|\; a(x),\, b(x)\in\mathbb{F}[[x]],\, b(x)\neq 0\right\},$$

with

$$\frac{a(x)}{b(x)}=\frac{c(x)}{d(x)}, \qquad \text{if} \qquad a(x)d(x)=b(x)c(x),$$

and with operations given by

$$\frac{a(x)}{b(x)} + \frac{c(x)}{d(x)} = \frac{a(x)d(x) + b(x)c(x)}{b(x)d(x)} \qquad \text{and} \qquad \frac{a(x)}{b(x)} \cdot \frac{c(x)}{d(x)} = \frac{a(x)c(x)}{b(x)d(x)}.$$

Proposition 1.2

a. *The invertible elements of* $\mathbb{F}[x]$ *are invertible elements of* $\mathbb{F}$.

b. *The invertible elements of* $\mathbb{F}[[x]]$ *are* $a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{F}[[x]]$ *with* a_0 *invertible in* $\mathbb{F}$.

Corollary 1.3 $\mathbb{F}((x)) = \left\{ x^k p(x) \;\middle|\; k \in \mathbb{Z},\ p(x) \in \mathbb{F}[[x]],\ p_0 \neq 0 \right\} \cup \{0\}$.

Let $p(x) \in \mathbb{F}((x))$. The **order** $\nu(p(x))$ of $p(x) = \sum_{l \in \mathbb{Z}} p_l x^l$ is the minimal integer l such that $p_l \neq 0$.

HW: Show that the order function $\nu : \mathbb{F}((x)) \to \mathbb{Z}$ is a *normalised discrete valuation* (see [BouC] Ch. VI §3 no.6 def.3).

2. References

[BouC] N. Bourbaki, *Commutative algebra*, Masson, Hermann ??? MR???????? (20??e:20????)