Proposition

Let X and Y be topological spaces.

Let $f: X \to Y$ be a continuous function

Let $E \subseteq X$.

(a) If E is connected then $f(E)$ is connected.

(b) If E is compact then $f(E)$ is compact.

Proposition

Let $X = \mathbb{R}$! Let $E \subseteq X$.

(a) E is connected if and only if E is an interval.

(b) E is compact and connected if and only if there exist $m, M \in \mathbb{R}$ such that $E = [m, M]$.

Theorem (Intermediate value Theorem)

Let $f: [a,b] \to \mathbb{R}$ be continuous. Then there exist $m, M \in \mathbb{R}$ such that

$$f([a,b]) = [m, M].$$

Theorem (Rolle's theorem)

Let $f: [a,b] \to \mathbb{R}$ be continuous with $f(a) = f(b)$.
If $f': (a,b) \to \mathbb{R}$ exists then there exists $c \in (a,b)$ such that $f'(c) = 0$

Idea that makes it work:

If M is the maximum value of f and $f(c) = M$ then $f'(c) = 0$.

<u>Theorem</u> (The mean value theorem)

Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that $f':(a,b)\to\mathbb{R}$ exists. Then there exists $c\in(a,b)$ such that

$$f(b)=f(a)+f'(c)(b-a).$$

<u>Theorem</u> (Taylor's theorem)

Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(N)}:(a,b)\to\mathbb{R}$ exists. Then there exists $c\in(a,b)$ such that

$$f(b)=f(a)+f'(a)(b-a)+\frac{1}{2!}f''(a)(b-a)^2$$
$$+\frac{1}{3!}f^{(3)}(a)(b-a)^2+\dots+\frac{1}{(N-1)!}f^{(N-1)}(a)(b-a)^{N-1}$$
$$+\frac{1}{N!}f^{(N)}(c)(b-a)^N.$$

If we ~~remove~~ let $a=0$ and $b=x$ then Taylor's theorem says that

$f(x)$ can be expanded as a <u>polynomial in x</u>,

$$f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\frac{1}{3!}f^{(3)}(0)x^3$$
$$+\frac{1}{4!}f^{(4)}(0)x^4+\dots$$

You need more and more terms for more and more accuracy in the expansion.

Just like...

You need more and more digits

$\pi = 3.141592\ldots$

for more and more accuracy in $\mathbb{R}$.

When does an infinite polynomial like

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

produce a good function? i.e.

For which $a \in \mathbb{C}$ does the series $\sum_{n=0}^{\infty} x^n$ converge.

In otherwords:

Let $s_1 = 1$, $s_2 = 1+a$, $s_3 = 1+a+a^2$; $s_4 = 1+a+a^2+a^3$, ...

For which $a \in \mathbb{C}$ does the sequence $(1, s_1, s_2, s_3, \ldots)$ converge?

In otherwords:

Does $\lim_{n\to\infty} s_n$ exist?

In otherwords:

Does there exist $l \in \mathbb{R}$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists ~~$\delta \in \mathbb{R}_{>0}$ such that~~

~~if $x \in \mathbb{A}$ and~~ $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(s_n, l) < \varepsilon$.

In this example:

$$s_1 = 1$$

$$s_2 = 1 + a$$

$$s_3 = 1 + a + a^2 = \frac{1-a^3}{1-a}$$

$$s_4 = 1 + a + a^2 + a^3 = \frac{1-a^4}{1-a}$$

$$\vdots$$

$$s_n = 1 + a + a^2 + \cdots + a^{n-1} = \frac{1-a^n}{1-a}.$$

So

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-a^n}{1-a}$$

$$= \frac{1}{1-a} \lim_{n\to\infty} 1 - a^n$$

$$= \frac{1}{1-a}\left(1 - \lim_{n\to\infty} a^n\right),$$

where we have used the limit theorems:

Theorem Let (a_n) and (b_n) be sequences (in $\mathbb{R}$ or in $\mathbb{R}^n$ ~~any metric space X~~) and assume $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Then

(a) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$,

(b) If $c \in \mathbb{R}$ then $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$,

(c) $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

(d) If ~~(b_n) seq~~ $b_n \neq 0$ for all $n \in \mathbb{Z}_{>0}$ then

$$\lim_{n\to\infty} \left(a_n / b_n\right) = \left(\lim_{n\to\infty} a_n\right) / \left(\lim b_n\right)$$

Skills:

(1) Algebraic manipulations - Expressions

(2) Graphing.

(3) Limits

(4) Sequences (functions $\mathbb{Z}_{>0} \to \mathbb{R}$ and $\mathbb{Z}_{>0} \to \mathbb{C}$)

(5) Series (a special kind of sequence)

(6) Approximations - Taylor's theorem
and Integral approximations

(7) Mathematical Language and Proof machine:
Fields, Ordered fields, Open sets
Inductions, Definitions & Proof machine.