620-295 Real Analysis with Applications, some additional material.

Proposition Let $X = \mathbb{R}$. Let $E \subseteq \mathbb{R}$. Then

E is connected if and only if E is an interval.

Proof To show: E is not connected if and only if E is not an interval.

To show: (a) If E is not connected then E is not an interval.

(b) If E is not an interval then E is not connected

(a) Assume E is not connected.

Then there exist open sets U_1, U_2 such that

$$U_1 \cup U_2 \supseteq E,\quad U_1 \cap E \neq \emptyset,\quad U_2 \cap E \neq \emptyset$$

and $(U_1 \cap E) \cap (U_2 \cap E) = \emptyset$.

Let $x \in U_1 \cap E$ and $y \in U_2 \cap E$ and suppose $x < y$.

Let

$$z = \cancel{\sup(U_1 \cap [x,y])}\ \sup(U_1 \cap E \cap [x,y])$$

Then $x \leq z \leq y$...

(b) To show: If E is not an interval then E is not connected.

Assume E is not an interval.

Then there exist $x, y \in E$ with $x < y$ and $z \notin E$ with $x < z < y$.

Let $U_1 = (-\infty, z)$ and $U_2 = (z, \infty)$

Then U_1, U_2 are open, $U_1 \cup U_2 \supseteq E$

$x \in U_1 \cap E$, $y \in U_2 \cap E$ and $(U_1 \cap E) \cap (U_2 \cap E) = \emptyset$.

So E is not connected//.

A __Hausdorff space__ is a topological space X such that

if $x, y \in X$ and $x \neq y$ then there exists a neighborhood N_x of x and a neighborhood N_y of y such that $N_x \cap N_y = \emptyset$.

The notes for Lectures 29 and 30 have a slight error in the definition of __not connected__.

The correct definition is:

Let X be a topological space.

Let $E \subseteq X$.

The set E is __not connected__ if there exist open sets U_1 and U_2 such that

$U_1 \cup U_2 \supseteq E$, $U_1 \cap E \neq \emptyset$, $U_2 \cap E \neq \emptyset$, and $(U_1 \cap E) \cap (U_2 \cap E) = \emptyset$.