Problem Set -- Theorems
620-295 Semester I 2010

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(1) Theorems

1. Theorems

(1) Prove that if (a_n) is a sequence in $\mathbb{C}$ and (a_n) converges then $\lim_{n\to\infty} a_n$ is unique.

(2) Prove that if (a_n) is a sequence in $\mathbb{C}$ and (a_n) converges then (a_n) is bounded.

(3) Prove that if (a_n) and (b_n) are sequences in $\mathbb{C}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n + b_n = a + b$.

(4) Prove that if (a_n) and (b_n) are sequences in $\mathbb{C}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n b_n = ab$.

(5) Prove that if (a_n) and (b_n) are sequences in $\mathbb{C}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ and $b_n \neq 0$ for all $n \in \mathbb{Z}_{>0}$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

(6) Prove that if (a_n), (b_n) and (c_n) are sequences in $\mathbb{R}$ and $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} c_n = \ell$ and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{Z}_{>0}$ then $\lim_{n\to\infty} b_n = \ell$.

(7) Prove that if (a_n) is a sequence in $\mathbb{R}$ and (a_n) is increasing and bounded above then (a_n) converges.

(8) Prove that if (a_n) is a sequence in $\mathbb{R}$ and (a_n) is not bounded then (a_n) diverges.

(9) Prove that if (a_n) is a contractive sequence in $\mathbb{R}$ then (a_n) is Cauchy. Give an example to show that the converse is not true.

(10) Prove that if (a_n) is a Cauchy sequence in $\mathbb{R}$ then (a_n) converges.

(11) Give an example of a Cauchy sequence (a_n) in $\mathbb{Q}$ that does not converge.

(12) Let X be a metric space. Prove that if (a_n) is a convergent sequence in X then (a_n) is Cauchy.

(13) Find a sequence a_n in $\mathbb{R}$ such that if $r \in [0, 1]$ there is a subsequence of a_n which converges to r.

(14) Let X be a metric space. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions and let $a \in X$. Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Show that $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

(15) Let X be a metric space. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions and let $a \in X$. Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Show that if $c \in \mathbb{R}$ then $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$,

(16) Let X be a metric space. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions and let $a \in X$. Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Show that $\lim_{x \to a} f(x)g(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

(17) Let X be a metric space. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions and let $a \in X$. Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Show that f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

(18) Assume that $\lim_{x \to a} g(x) = l$ and $\lim_{y \to l} f(y)$ exists. Show that $\lim_{y \to l} f(y) = \lim_{x \to a} f(g(x))$.

(19) Let a_n and b_n be sequences in $\mathbb{R}$. Assume that $\lim_{n \to \infty} a_n$ exists and $\lim_{n \to \infty} b_n$ exists. Show that if $a_n \leq b_n$ then $\lim_{n \to \infty} a_n \leq b_n$.

(20) Let $x \in \mathbb{C}$. Show that

$$\lim_{n \to \infty} x^n = \begin{cases} 0, & \text{if} \quad |x| < 1, \\ \text{diverges}, & \text{if} \quad |x| > 1, \\ 1, & \text{if} \quad x = 1, \\ \text{diverges}, & \text{if} \quad |x| = 1 \quad \text{and} \quad x \neq 1. \end{cases}$$

(21) Let $x \in \mathbb{C}$. Prove that

$$\lim_{n \to \infty} 1 + x + x^2 + \ldots + x^n = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \dfrac{1}{1 - x}, & \text{if} \quad |x| < 1, \\ \text{diverges}, & \text{if} \quad |x| \geq 1. \end{cases}$$

(22) Let $k \in \mathbb{R}_{>0}$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \quad \text{converges if} \quad k > 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^k} \quad \text{diverges if} \quad k \le 1.$$

(23) Prove that $\lim_{x \to 0} e^x = e^0$.

(24) Let $a \in \mathbb{C}$. Prove that $\lim_{x \to a} e^x = e^a$.

(25) Let $n \in \mathbb{Z}_{>0}$ and $a \in \mathbb{C}$. Show that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x) = x^n$ is continuous at $x = a$.

(26) Let $a \in \mathbb{C}$. Show that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(x) = e^x$ is continuous at $x = a$.

(27) Let $r, s \in \mathbb{C}$. Assume $\sum_{n=0}^{\infty} a_n s^n$ converges. Show that if $|r| < |s|$ then $\sum_{n=0}^{\infty} a_n |r|^n$ converges.

(28) Let (a_n) be a sequence in $\mathbb{C}$. Show that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(29) Let (a_n) be a sequence in $\mathbb{R}$ such that $a_n \in \mathbb{R}_{\ge 0}$, if $n \in \mathbb{Z}_{>0}$ then $a_n \ge a_{n+1}$, and $\lim_{n \to \infty} a_n = 0$. Show that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

(30) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ and let $\beta, \gamma \in \mathbb{R}$. Let $c \in [a, b]$ and assume that $f'(c)$ and $g'(c)$ exist. Prove that $(\beta f + \gamma g)'(c) = \beta f'(c) + \gamma g'(c)$.

(31) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. Assume that $f'(c)$ and $g'(c)$ exist. Prove that $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(32) Let $f : [a, b] \to \mathbb{R}$ be given by $f(x) = x$ and let $c \in [a, b]$. Prove that $f'(c) = 1$.

(33) Let $f : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. Prove that if $f'(c)$ exists then f is continuous at $x = c$.

(34) Prove that $\exp'(x) = \exp(x)$.

(35) Carefully state and prove the intermediate value theorem.

(36) Carefully state and prove the max-min theorem.

(37) Carefully state and prove Rolle's theorem.

(38) Carefully state and prove the mean value theorem.

(39) Carefully state and prove Taylor's theorem with Lagrange's remainder.

(40) Carefully state and prove the fundamental theorem of calculus.

(41) Carefully state the trapezoidal integral approximation and the bound on the error.

(42) Carefully state Simpson's integral approximation and the bound on the error.

(43) Prove that $\mathbb{Q}$ is a field.

(44) Prove that $\mathbb{R}$ is a field.

(45) Prove that $\mathbb{C}$ is a field.

(46) Prove that $\mathbb{Q}$ is an ordered field.

(47) Prove that $\mathbb{R}$ is an ordered field.

(48) Prove that $\mathbb{C}$ is not an ordered field.

(49) Carefully state and prove Lagrange's identity.

(50) Carefully state and prove the Schwarz inequality.

(51) Carefully state and prove the triangle inequality.

(52) Prove that $\mathbb{R}$ is a metric space.

(53) Prove that $\mathbb{R}^n$ is a metric space.

(54) Prove that $\mathbb{R}$ is a topological space.

(55) Prove that $\mathbb{R}^n$ is a topological space.

(56) Let X be a metric space. Prove that X is a topological space.

(57) Let $E \subseteq \mathbb{R}$. Show that E is connected if and only if E is an interval.

(58) Let $E \subseteq \mathbb{R}$. Show that E is connected and compact if and only if E is a closed and bounded interval.

(59) Let S be a set with a partial order and let E be a subset of S. Show that if $\sup E$ exists then it is unique.