Real Analysis with Applications 620-295, Lect. 30, 19 May 2010 (1)

Let X be a topological space with topology $\mathcal{T}$.

Let $E \subseteq X$.

In English: The interior of E is the largest open set contained in E.

In maths:

The interior of E is a set E° such that

(a) E° is open and $E^\circ \subseteq E$,

(b) If U is open and $U \subseteq E$ then $U \subseteq E^\circ$.

Let $p \in X$.

A neighborhood of p is an open set U such that $p \in U$.

Let $E \subseteq X$.

A interior point of E is a $p \in E$ such that there exists a neighborhood U of p with $U \subseteq E$.

Theorem Let X be a topological space. Let $E \subseteq X$. Then $E^\circ = \{p \in E \mid p \text{ is an interior point of } E\}$.

Proof Let $F = \{p \in E \mid p \text{ is an interior point of } E\}$.

To show: $E^\circ = F$.

To show: (a) $E^\circ \subseteq F$

(b) $F \subseteq E^\circ$

(a) To show: If $p \in E^\circ$ then p is an interior point of E.

Assume $p \in E^\circ$.

To show: p is an interior point of E.

Since E° is open and $p \in E^\circ$ and $E^\circ \subseteq E$,

p is an interior point of E.

(b) To show: If p is an interior point of E then $p \in E^\circ$.

Assume p is an interior point of E

To show: $p \in E^\circ$.

There is a neighborhood U of p with $U \subseteq E$.

Since U is open and $U \subseteq E$ then $U \subseteq E^\circ$.

So $p \in E^\circ$, because $p \in U$. //.

A function $f:[a,b]\to\mathbb{R}$ is normal continuous if f satisfies:

If $c\in[a,b]$ then $\lim_{x\to c} f(x)=f(c)$

A function $f:[a,b]\to\mathbb{R}$ is topology continuous if f satisfies

If $V\subseteq\mathbb{R}$ is open then $f^{-1}(V)$ is open

Theorem Let $f:[a,b]\to\mathbb{R}$ be a function.
f is normal continuous if and only if
f is topology continuous.

Proof To show: (a) If f is normal continuous then f is topology continuous

(b) If f is topology continuous then f is normal continuous.

(a) Assume f is normal continuous.

To show: f is topology continuous.

To show: If $V\subseteq\mathbb{R}$ is open then $f^{-1}(V)$ is open.

Assume $V\subseteq\mathbb{R}$ is open.

To show: $f^{-1}(V)$ is open.

To show: If $p\in f^{-1}(V)$ then p is an interior point of $f^{-1}(V)$.

Assume $p \in f^{-1}(V)$.

To show: p is an interior point of $f^{-1}(V)$.

We know $f(p) \in V$.

Since V is open, $f(p)$ is an interior point of V.

So there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(f(p)) \subseteq V$.

Since f is normal continuous there exists $\delta \in \mathbb{R}_{>0}$ such that if $d(x,p) < \delta$ then $d(f(x), f(p)) < \varepsilon$.

So there exists $\delta \in \mathbb{R}_{>0}$ such that $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$

So $f(B_\delta(p)) \subseteq V$.

So $B_\delta(p) \subseteq f^{-1}(V)$.

So p is an interior point of $f^{-1}(V)$.

(b) Assume f is topology continuous.

To show: f is normal continuous

To show: If $p \in [a,b]$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $d(x,p) < \delta$ then $d(f(x), f(p)) < \varepsilon$.

Assume $p \in [a,b]$ and $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that

$$f(B_\delta(p)) \subseteq B_\varepsilon(f(p)).$$

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that

$$B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p)))$$

Since f is topology continuous and $B_\varepsilon(f(p))$ is open then $f^{-1}(B_\varepsilon(f(p)))$ is open.

So p is an interior point of $f^{-1}(B_\varepsilon(f(p)))$

So there exists $B_\delta(p)$ with $B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p)))$.

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