Problem Set -- Topology
620-295 Semester I 2010

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1. Topology

(1) Define topology and topological space.

(2) Let X be a set with 2 points. Find all topologies on X.

(3) Let X be a set with 3 points. Find all topologies on X.

(4) Define open, closed, connected and compact sets.

(5) Define complement, open cover, subcover and finite subcover.

(6) Let X and Y be topological spaces. Define continuous function $f: X \to Y$.

(7) Let X and Y be topological spaces. Let $f: X \to Y$ be a continuous function and let E be a connected subset of X. Show that $f(E)$ is connected.

(8) Let X and Y be topological spaces. Let $f: X \to Y$ be a continuous function and let E be a compact subset of X. Show that $f(E)$ is compact.

(9) Define neighborhood of a point p, interior point of E, and close point to E.

(10) Let X be a topological space and let E be a subset of X. Define the interior and closure of E.

(11) Let X be a topological space and let E be a subset of X. Prove that E° is the set of interior points of E.

(12) Let X be a topological space and let E be a subset of X. Prove that $\overline{E}$ is the set of close points of E.

(13) Let X be a topological space and let E be a subset of X. Prove that E is open if and only if $E = E^{\circ}$.

(14) Let X be a topological space and let E be a subset of X. Prove that E is closed if and only if $E = \overline{E}$.

(15) Define the standard topology on a metric space.

(16) Let X be a topological space. Define Hausdorff.

(17) Let X be a Hausdorff topological space. Let E be a subset of X. Show that if E is compact then E is closed. Give an example to show that the converse does not hold.

(18) Let X be a metric space. Show that X is Hausdorff.

(19) Let X be a metric space. Let E be a subset of X. Show that if E is compact then E is closed and bounded. Give an example to show that the converse does not hold.

(20) Let E be a subset of $\mathbb{R}^n$ (with the standard topology). Show that E is compact if and only if E is closed and bounded.

(21) Prove that the connected sets in $\mathbb{R}$ are the intervals.

(22) Prove that if E is a connected compact set in $\mathbb{R}$ then there exist $m, M \in \mathbb{R}$ such that $E = [m, M]$.

(23) Define a topology on a partially ordered set X. Are the connected sets in X only the intervals?

(24) Show that the set $[1, 2]_{\mathbb{Q}} = \{x \in \mathbb{Q} \mid 1 \leq x \leq 2\}$ is not a connected subset of $\mathbb{Q}$.

(25) Show that the set $[1, 2]_{\mathbb{R}} = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$ is a connected subset of $\mathbb{R}$.

(26) Give an example of a subset of $\mathbb{R}$ that is both open and closed.

(27) Give an example of a subset of $\mathbb{R}$ that is not open and not closed.

(28) Give an example of a subset of $\mathbb{R}$ that is open and not closed.

(29) Give an example of a subset of $\mathbb{R}$ that is closed and not open.

2. Definitions

(1) Define set, union, intersection and product (of sets).

(2) Define set, subset, and equal sets.

(3) Define function, injective, surjective and bijective.

(4) Define cardinality, finite, infinite, countable and uncountable.

(5) Define function, composition of functions, equal functions, identity function.

(6) Define the positive integers, addition, multiplication and the order on positive integers.

(7) Let S be a set. Define (a) relation on S and (b) operation on S.

(8) Define field.

(9) Let $z \in \mathbb{C}$. Define the conjugate of z.

(10) Define the absolute value on $\mathbb{R}$.

(11) Define the absolute value on $\mathbb{C}$.

(12) Define the absolute value on $\mathbb{R}^{\mathrm{n}}$.

(13) Define partial order and total order.

(14) Define ordered field.

(15) Define e^x and $\log x$.

(16) Define $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, and $\cot x$.

(17) Define $\sinh x$, $\cosh x$, $\tanh x$, $\mathrm{sech}\, x$, $\mathrm{csch}\, x$, and $\coth x$.

(18) Define $\arcsin x$, $\arccos x$, $\arctan x$, $\mathrm{arcsec}\, x$, $\mathrm{arccsc}\, x$, and $\mathrm{arccot}\, x$.

(19) Define $\mathrm{arcsinh}\, x$, $\mathrm{arccosh}\, x$, $\mathrm{arctanh}\, x$, $\mathrm{arcsech}\, x$, $\mathrm{arccsch}\, x$, and $\mathrm{arccoth}\, x$.

(20) Define $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}$.

(21) Define $\mathbb{Q}[x]$ and $\mathbb{Q}[[x]]$.

(22) Define the addition and the multiplication on $\mathbb{Q}[x]$ and $\mathbb{Q}[[x]]$.

(23) Define $\mathbb{Q}$, $\mathbb{Q}(x)$ and $\mathbb{Q}((x))$.

(24) Define the addition and the mutliplication on $\mathbb{Q}$, $\mathbb{Q}(x)$ and $\mathbb{Q}((x))$.

(25) Define $\mathbb{R}$.

(26) Define $\mathbb{R}^n$.

(27) Define the addition, the multiplication and the order on $\mathbb{R}$.

(28) Define the the absolute value, the distance and the topology on $\mathbb{R}$.

(29) Define $\mathbb{C}$.

(30) Define the addition, the multiplication, the absolute value, the distance and the topology on $\mathbb{C}$.

(31) Define sequence, bounded, increasing, decreasing and monotone.

(32) Define sequence, convergent, divergent, Cauchy and contractive.

(33) Define series, convergent series, and divergent series.

(34) Define series, absolutely convergent series, and conditionally convergent series.

(35) Define series, geometric series, p-harmonic series, and the Riemann zeta function.

(36) Define sequence, $\sup a_n$, $\inf a_n$, $\limsup a_n$, and $\liminf a_n$.

(37) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define what it means for f to be continuous.

(38) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $p \in [a, b]$. Define what it means for f to be continuous at p.

(39) Let X and Y be metric spaces and let $f : X \to Y$ be a function. Define what it means for f to be continuous.

(40) Let X and Y be metric spaces and let $f : X \to Y$ be a function. Let $p \in X$. Define what it means for f to be continuous at p.

(41) Let (a_n) be a sequence in $\mathbb{R}$. Define $\lim_{n\to\infty} a_n$.

(42) Let X be a metric space and let (a_n) be a sequence in X. Define $\lim_{n\to\infty} a_n$.

(43) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define $\lim_{x\to a} f(x)$.

(44) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $p \in [a, b]$. Define $\lim_{x\to p} f(x)$.

(45) Let X and Y be metric spaces and let $f : X \to Y$ be a function. Let $p \in X$. Define $\lim_{x\to p} f(x)$.

(46) Define $\int_a^\infty f(x)dx$.

(47) Define $\int_{-\infty}^b f(x)dx$.

(48) Define $\int_a^b f(x)dx$ when the integral is improper.

(49) Define improper integrals.

(50) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $c \in [a, b]$. Define $f'(c)$.

(51) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define what it means for f to be differentiable.

(52) Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $p \in [a, b]$. Define what it means for f to be differentiable at p.

(53) Define series expansion of $f(x)$ at $x = a$.

(54) Define Taylor approximation of $f(x)$ at $x = a$ of order n.

(55) Define Lagrange's remainder.

(56) Define Riemann integral.

(57) Define trapezoidal approximation.

(58) Define Simpson approximation.

3. Examples

(1) Define partial order Give an example of (a) a total order and (b) a partial order that is not a total order.

(2) Give an example of (a) an ordered field and (b) a field that is not an ordered field.

(3) Give an example of a sequence of rational numbers that is Cauchy but does not converge.

(4) Give an example of a subset of $\mathbb{Q}$ that has an upper bound but does not have a least upper bound.

(5) Give an example of an element of $\mathbb{Q}$ that is not an element of $\mathbb{R}$.

(6) Give an example of a sequence of real numbers such that $\sup a_n$, $\inf a_n$, $\limsup a_n$, and $\liminf a_n$ are all different.

(7) Give an example of a series that is conditionally convergent but not absolutely convergent.