620-295 Real Analysis with Applications, Lect 29, 17 May 2010

Topology:

Goal: Understand and prove the

Intermediate value theorem and

The max-min theorem.

Theorem (Intermediate Value Theorem)
Let $f: [a,b] \to \mathbb{R}$ be a continuous function
If $x \in \mathbb{R}$ and x is between $f(a)$ and $f(b)$
then there exists $c \in [a,b]$ with $f(c) = x$.

Theorem (Max-Min theorem).
Let $f: [a,b] \to \mathbb{R}$ be a continuous function.
Then there exists $m, M \in \mathbb{R}$ such that
$$f([a,b]) = [m, M].$$

Let X be a set

A topology on X is a collection $\mathcal{T}$ of subsets of X such that

(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\bigcup_{u \in \mathcal{S}} u \in \mathcal{T}$

(b) If $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{T}$ then
$U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

A topological space is a set X with a topology on X.

Let X be a topological space with topology $\mathcal{T}$.
Let E be a subset of X.
The set E is <u>open</u> if $E \in \mathcal{T}$.

<u>Examples</u> Let $X = \{a, b, c, d\}$.

$$\mathcal{T} = \left\{ \begin{array}{l} \phi, \{a\}, \{b,c\}\ \{a,b,c\} \\ \quad \{d\}, \ \{\cancel{b},c,d\}, \\ \{a,b,c,d\} \end{array} \right\}$$
is not a topology on X.

$$\mathcal{T} = \left\{ \begin{array}{l} \phi\ \ \{a\}, \{d\}, \ \{b,c\}, \{a,d\} \\ \quad \{a,b,c\}, \ \{b,c,d\} \\ \quad \{a,b,c,d\} \end{array} \right\}$$
is a topology on X.

Let $X = \mathbb{R}^n$.

$\mathcal{T} = \{$ open subsets $E \subseteq \mathbb{R}^n \}$

is a topology on $\mathbb{R}^n$ (a subset $E \subseteq \mathbb{R}^n$ is <u>open</u> if E is a union of ε-balls).

Let X be a topological space with topology $\mathcal{T}$.

Let E be a subset of X.

The set E is <u>closed</u> if E^c is open

Recall: $E^c = \{ x \in X \mid x \notin E \}$.

The set E is <u>connected</u> if there do not exist open sets U_1, U_2 such that

$$U_1 \cup U_2 \supseteq E \quad \text{and} \quad (U_1 \cap E) \cap (U_2 \cap E) = \emptyset.$$

The set E is <u>compact</u> if E satisfies:

If $\mathcal{S} \subseteq \mathcal{T}$ and $\left(\bigcup_{U \in \mathcal{S}} U \right) \supseteq E$

then there exists $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{S}$

such that $(U_1 \cup U_2 \cup \cdots \cup U_n) \supseteq E$.

In English: E is compact if every open cover of E has a finite subcover.

What does this have to do with the Max-min theorem?

<u>Theorem</u> Let $X = \mathbb{R}$ with the standard topology. Let $E \subseteq \mathbb{R}$. Then

E is compact and connected

if and only if

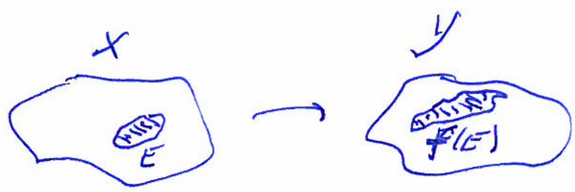
there exist $m, M \in \mathbb{R}$ such that $E = [m, M]$

Continuous functions for topological spaces

Let X and Y be sets.

Let $f: X \to Y$ be a function.

Let E be a subset of X.

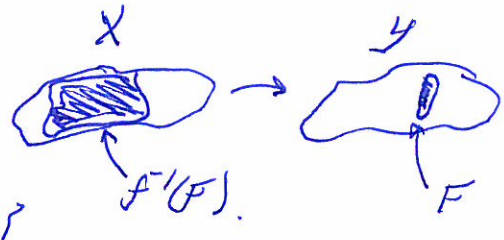


The image of E is

$$f(E) = \{ f(x) \mid x \in E \}$$

Let F be a subset of Y.

The inverse image of F is



$$f^{-1}(F) = \{ x \in X \mid f(x) \in F \}$$

Warning: Despite the confusing notation $f^{-1}(F)$ has nothing to do with the inverse function to f.

Let X be a topological space with topology $\mathcal{T}$

Let Y be a topological space with topology $\mathcal{Z}$

Let $f: X \to Y$ be a function.

The function $f: X \to Y$ is continuous if f satisfies:

If $V \in \mathcal{Z}$ then $f^{-1}(V) \in \mathcal{T}$

In English: Inverse images of open sets are open.

Let $a, b \in \mathbb{R}$ with $a < b$

Let $X = [a,b]$ with the topology given by

$E \subseteq X$ is open if E is a union of ε-balls.

Let $Y = \mathbb{R}$ with the standard topology.

<u>Theorem</u> Let $f : [a,b] \to \mathbb{R}$. The function f is continuous (as a function between topological spaces) if and only if f satisfies

if $c \in [a,b]$ then $\lim_{x \to c} f(x) = f(c)$.