620-295 Real Analysis with applications Lect. 28, 14 May 2010 ①

The real numbers $\mathbb{R}$

The real numbers $\mathbb{R}$ is the set

$$\mathbb{R} = \mathbb{R}_{\le 0} \cup \mathbb{R}_{\ge 0}, \text{ where}$$

$$\mathbb{R}_{\ge 0} = \{a_\ell a_{\ell-1} \cdots a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots \mid \ell \in \mathbb{Z}_{\ge 0},\ a_i \in \{0,1,\ldots,9\}\}$$

$$\mathbb{R}_{\le 0} = \{-a_\ell a_{\ell-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \mid \ell \in \mathbb{Z}_{\ge 0},\ a_i \in \{0,1,\ldots,9\}\}$$

and

$$a = b \quad \text{if} \quad a - b = 0.000\ldots.$$

Define a relation $\le$ on $\mathbb{R}$ by

$$x \le y \quad \text{if} \quad y - x \in \mathbb{R}_{\ge 0}.$$

Define the absolute value on $\mathbb{R}$

$$\begin{aligned} |\cdot| : \mathbb{R} &\to \mathbb{R}_{\ge 0} \\ x &\mapsto |x| \end{aligned} \quad \text{by} \quad |x| = \begin{cases} x, & \text{if } x \in \mathbb{R}_{>0} \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x \in \mathbb{R}_{<0} \end{cases}$$

Define a distance on $\mathbb{R}$, $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, by

$$d(x,y) = |y - x|$$

Let $\varepsilon \in \mathbb{R}_{>0}$. ~~The~~ and $x \in \mathbb{R}$. The ε-ball at x is

$$B_\varepsilon(x) = \{y \in \mathbb{R} \mid d(x,y) < \varepsilon\}$$

Let E be a subset of $\mathbb{R}$. The set E is open if

E is a union of ε-balls

The Euclidean space $\mathbb{R}^n$

The <u>Euclidean space</u> $\mathbb{R}^n$ is the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}.$$

Define the <u>absolute value</u> on $\mathbb{R}^n$,

$$\begin{array}{rcl} |\cdot| : \mathbb{R}^n & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & |x| \end{array} \quad \text{by} \quad |x| = \sup\left(\pm\sqrt{x_1^2 + \cdots + x_n^2}\right)$$

if $x = (x_1, \ldots, x_n)$.

The <u>distance</u> on $\mathbb{R}^n$ is $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$d(x,y) = |y - x|.$$

Let $\varepsilon \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^n$. The <u>ε-ball at x</u> is

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x,y) < \varepsilon\}$$

Let E be a subset of $\mathbb{R}^n$. The set E is <u>open</u> if E is a union of ε-balls.

A <u>metric space</u> is a set X with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $d(x,x) = 0$,

(b) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$,

(c) If $x, y \in X$ then $d(x,y) = d(y,x)$,

(d) If $x, y, z \in X$ then $d(x,z) \leq d(x,y) + d(y,z)$

Let X be a metric space. Let $x \in X$.
Let $\varepsilon \in \mathbb{R}_{>0}$. The <u>$\varepsilon$-ball at X</u> is

$$B_\varepsilon(X) = \{ y \in X \mid d(y,x) < \varepsilon \}$$

<u>Example</u> Let $C = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and bounded} \}$

Define $\| \; \| : C \to \mathbb{R}_{\geq 0}$ by

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$$

Define a distance $d : C \times C \to \mathbb{R}_{\geq 0}$ by

$$d(f,g) = |g-f|$$

where $(g-f)(x) = g(x) - f(x)$, for $x \in \mathbb{R}$.
Then C is a metric space. //.

Let X be a metric space and let E be a subset of X.
The set E is <u>open</u> if E is a union of ε-balls.

<u>Example</u> $E = \{ x \in \mathbb{R} \mid |x+2| \leq 2 \text{ or } |x| > 1 \}$

So $E = \{ x \in \mathbb{R} \mid |x+2| \leq 2 \} \cup \{ x \in \mathbb{R} \mid |x| > 1 \}$

$$= \{ x \in \mathbb{R} \mid x \geq -2 \text{ and } x+2 \leq 2 \}$$
$$\cup \{ x \in \mathbb{R} \mid x < -2 \text{ and } -(x+2) \leq 2 \}$$
$$\cup \{ x \in \mathbb{R} \mid x > 1 \} \cup \{ x \in \mathbb{R} \mid x < -1 \}$$

$$= \{x \in \mathbb{R} \mid x \geq -2 \text{ and } x \leq 0\}$$
$$\cup \{x \in \mathbb{R} \mid x < -2 \text{ and } x+2 \geq -2\}$$
$$\cup (1, \infty) \cup (-\infty, -1)$$

$$= [-2, 0] \cup \{x \in \mathbb{R} \mid x < -2 \text{ and } x \geq -4\}$$
$$\cup (1, \infty) \cup (-\infty, -1)$$

$$= [-2, 0] \cup [-4, -2) \cup (1, \infty) \cup (-\infty, -1)$$

So $E = (-\infty, 0] \cup (1, \infty)$.

E is <u>not</u> open in $\mathbb{R}$.

E is not bounded above and not bounded below.