620-295 Real Analysis with applications, Lect 27 ①

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Let S be a set.

A relation on S is a subset Γ of $S \times S$.

Example $S = \{\alpha, \beta, \gamma\}$

$$S \times S = \begin{Bmatrix} (\alpha,\alpha), (\alpha,\beta), (\alpha,\gamma) \\ (\beta,\alpha), (\beta,\beta), (\beta,\gamma) \\ (\gamma,\alpha), (\gamma,\beta), (\gamma,\gamma) \end{Bmatrix} \qquad \Gamma = \{(\beta,\beta), (\beta,\gamma), (\gamma,\alpha)\}.$$

A partial order on S is a relation Γ on S such that

(a) If $x, y, z \in S$ and $(x,y) \in \Gamma$ and $(y,z) \in \Gamma$ then $(x,z) \in \Gamma$

(b) If $x, y \in S$ and $(x,y) \in \Gamma$ and $(y,x) \in \Gamma$ then $x = y$.

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(a) If $x, y, z \in S$ and $(x,y) \in \Gamma$ and $(y,z) \in \Gamma$ then $(x,z) \in \Gamma$,

(b) If $x, y \in S$ and $(x,y) \in \Gamma$ and $(y,x) \in \Gamma$ then $x = y$,

(c) If $x, y \in S$ then $(x,y) \in \Gamma$ or $(y,x) \in \Gamma$.

If Γ is a partial order on S write

$x \le y$ if $(x,y) \in \Gamma$.

<u>Example</u> (a partial order that is not a total order).

Let $E = \{\alpha, \beta, \gamma\}$ and let

$$S = \{\text{subsets of } E\} = \left\{ \begin{array}{c} \phi,\ \{\alpha\},\ \{\beta\},\ \{\gamma\} \\ \{\alpha,\beta\},\ \{\alpha,\gamma\},\ \{\beta,\gamma\} \\ \{\alpha,\beta,\gamma\} \end{array} \right\}$$

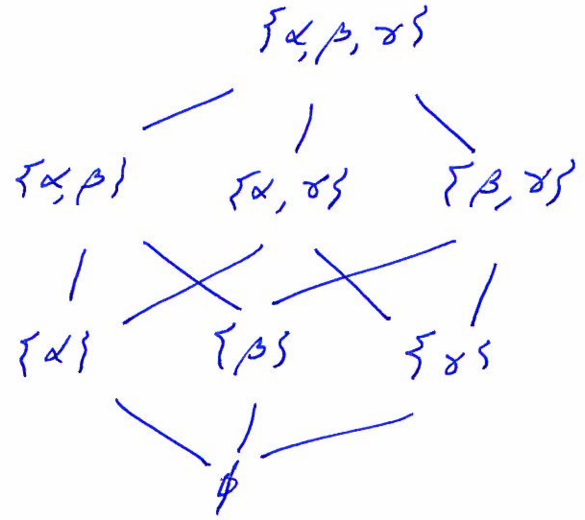
Let Γ be the relation on S given by

$$\Gamma = \{ (A,B) \mid A \subseteq B \}.$$

In otherwords <u>inclusion</u> is a ~~relation~~ partial order on S since

(a) If $A, B, C \in S$ and $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

(b) If $A, B \in S$ and $A \subseteq B$ and $B \subseteq A$ then $A = B$.



Inclusion is <u>not</u> a total order since

$X = \{\alpha\}$ and $Y = \{\beta\}$ are in S

and $X \not\subseteq Y$ and $Y \not\subseteq X$.

Let S be a set with a partial order Γ

Write $x \leq y$ if $(x,y) \in \Gamma$.

Let A be a subset of S.

An <u>upper</u> <u>bound</u> of A is an element $b \in S$ such that

if $a \in A$ then $a \leq b$.

A <u>lower</u> <u>bound</u> of A is an element $l \in S$ such that

if $a \in A$ then $l \leq a$

A <u>maximum</u> of A is an element $M \in A$ such that

there does not exist $a \in A$ such that $a \geq M$

(i.e. if $a \in A$ then $M \not\leq a$).

A <u>minimum</u> of A is an element $m \in A$ such that

if $a \in A$ then $m \not\geq a$.

A <u>supremum</u> of A is an element $s \in S$ such that

(a) s is an upper bound of A

(b) If b is an upper bound of A then $b \geq s$.

An <u>infimium</u> of A is an element $i \in S$ such that

(a) i is a lower bound of A,

(b) If l is a lower bound of A then $l \leq i$.

<u>Example</u> If $A = \{\{\alpha,\beta\}, \{\beta,\gamma\}, \{\beta\}\}$

then $\{\alpha,\beta\}$ and $\{\beta,\gamma\}$ are both maximums of A

and $\sup A = \{\alpha,\beta,\gamma\}$.

<u>Example</u> Prove that $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}((0,1]_{\mathbb{R}})$

where $(0,1]_{\mathbb{R}} = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$.

<u>Proof</u>

Proof by contradiction.

Assume $f: \mathbb{Z}_{>0} \longrightarrow (0,1]_{\mathbb{R}}$ is a bijection
$k \longmapsto r_k$

$r_1 = 0.r_{11}r_{12}r_{13}r_{14}\ldots$

$r_2 = 0.r_{21}r_{22}r_{23}r_{24}\ldots$

$r_3 = 0.r_{31}r_{32}r_{33}r_{34}\ldots$

$r_4 = 0.r_{41}r_{42}r_{43}r_{44}\ldots$

$\vdots$

Let $s = 0.s_1s_2s_3s_4s_5s_6$ with

$s_1 \neq r_{11}$, $s_2 \neq r_{22}$, $s_3 \neq r_{33}$, ...

Then $s \in (0,1]_{\mathbb{R}}$ and does not appear in the sequence $(r_1, r_2, r_3, \ldots)$

So f is not surjective. This is a contradiction to f being bijective. So $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}((0,1]_{\mathbb{R}})$

Let S be a set and let Γ be a partial order on S. Write

$$x \le y \quad \text{if} \quad (x,y) \in \Gamma.$$

Let $a, b \in S$. Then let

$$[a,b] = \{x \in S \mid a \le x \le b\}$$

$$(a,b] = \{x \in S \mid a < x \text{ and } x \le b\}$$

$$[a,b) = \{x \in S \mid a \le x \text{ and } x < b\}$$

$$(a,b) = \{x \in S \mid a < x \text{ and } x < b\}$$

$$[a,\infty) = \{x \in S \mid a \le x\}$$

$$(a,\infty) = \{x \in S \mid a < x\}$$

$$(-\infty,a] = \{x \in S \mid x \le a\}$$

$$(-\infty,a) = \{x \in S \mid x < a\}$$

where

$$x < y \text{ means } x \le y \text{ and } x \ne y.$$

The sets in (*) are <u>intervals</u> in S.