Sets

A set is a collection of elements.

Write $s \in S$ if s is an element of the set S

Let S and T be sets.

T is a subset of S if T satisfies:

if $t \in T$ then $t \in S$.

T is equal to S if $T \subseteq S$ and $S \subseteq T$.

The intersection of S and T is the set

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}.$$

The union of S and T is the set

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}.$$

The product of S and T is the set

$$S \times T = \{(s,t) \mid s \in S,\ t \in T\}$$

of pairs with the first entry from S and the second entry from T.

Example $S = \{1, 2, 3, 5, 6, 7\}$
$T = \{2, 3, 4, 6, 7, 8\}$

Then $S \cap T = \{2, 3, 6, 7\}$

$S \cup T = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$$S \times T = \left\{ \begin{matrix} (1,2), (1,3), (1,4), (1,6), (1,7), (1,8), \\ (2,2), (2,3), (2,4), (2,6), (2,7), (2,8), \\ (3,2), (3,3), \ldots \end{matrix} \right\}.$$

Functions

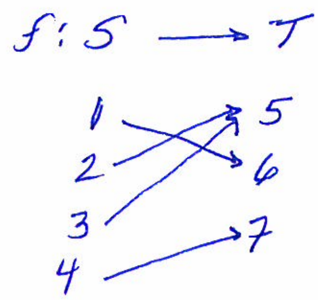
A <u>function</u> $f: S \to T$ is an assignment of an element $f(s) \in T$ to each $s \in S$.

A function $f: S \to T$ is <u>injective</u> if it satisfies: if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$.

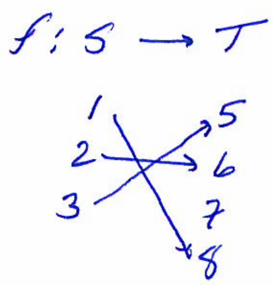
A function $f: S \to T$ is <u>surjective</u> if it satisfies: if $t \in T$ then there exists $s \in T$ such that $f(s) = t$.

A function $f: S \to T$ is <u>bijective</u> if it is injective and surjective.

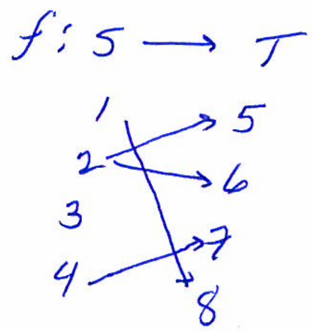
Examples



surjective
not injective



injective
not surjective



not a function

<u>Cardinality</u>

Let S and T be sets.

S and T have the same <u>cardinality</u> if there exists a bijective function $f: S \to T$.

Write $\mathrm{Card}(S) = \mathrm{Card}(T)$ if there exists a bijective function $f: S \to T$.

Let S be a set.

(a) S is <u>finite</u> if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Card}(S) = \mathrm{Card}(\{1, 2, \ldots, n\})$.

(b) S is <u>infinite</u> if S is not finite.

(c) S is <u>countable</u> if S is finite or $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z})$.

(d) S is <u>uncountable</u> if S is not countable.

Write $\mathrm{Card}(S) = n$ if $n \in \mathbb{Z}_{\geq 0}$ and $\mathrm{Card}(S) = \mathrm{Card}(\{1, 2, \ldots, n\})$.

④

Example Prove that $\mathrm{Card}(\mathbb{Z}_{>0}) = \mathrm{Card}(\mathbb{Z}_{\geq 0})$.

To show: There exists a bijective function $f: \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$.

Let

$$\begin{array}{rcl} f: \mathbb{Z}_{>0} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ 1 & \longmapsto & 0 \\ 2 & \longmapsto & 1 \\ 3 & \longmapsto & 2 \\ \vdots & & \vdots \end{array}$$

Example Prove that $\mathrm{Card}(\mathbb{Z}_{\geq 0}) = \mathrm{Card}(\mathbb{Z})$

To show: There exists a bijective function $f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}$.

Let

$$\begin{array}{rcl} f: \mathbb{Z}_{\geq 0} & \longrightarrow & \mathbb{Z} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & -1 \\ 3 & \longmapsto & 2 \\ 4 & \longmapsto & -2 \\ 5 & \longmapsto & 3 \\ 6 & \longmapsto & -3 \\ \vdots & & \vdots \end{array}$$

Example Let $[0,1]_{\mathbb{Q}} = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.

Show that $\mathrm{Card}([0,1]_{\mathbb{Q}}) = \mathrm{Card}(\mathbb{Z}_{>0})$

List the expressions $\frac{a}{b}$ with $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}_{>0}$ in the order

$$\left(\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \dots\right)$$

Take the subsequence of this sequence of reduced expression of elements in $(0,1]_{\mathbb{Q}}$,

$$\left(\frac{1}{1}, \frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \quad \frac{3}{4}, \quad \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots\right)$$

This sequence is a bijective function

$$\begin{array}{rcl} f: \mathbb{Z}_{>0} & \longrightarrow & (0,1]_{\mathbb{Q}} \\ 1 & \longmapsto & \frac{1}{1} \\ 2 & \longmapsto & \frac{1}{2} \\ 3 & \longmapsto & \frac{1}{3} \\ 4 & \longmapsto & \frac{2}{3} \\ 5 & \longmapsto & \frac{1}{4} \\ 6 & \longmapsto & \frac{3}{4} \\ 7 & \longmapsto & \frac{1}{5} \\ \vdots & & \vdots \end{array}$$