A *metric space* is a set X with a function
$d: X \times X \longrightarrow \mathbb{R}_{\geq 0}$ such that

(a) if $p \in X$ then $d(p,p) = 0$,

(b) if $p, q \in X$ and $p \neq q$ then $d(p,q) \neq 0$,

(c) if $p, q \in X$ then $d(p,q) = d(q,p)$,

(d) If $p, q, r \in X$ then $d(p,r) \leq d(p,q) + d(q,r)$.

The point of this lecture is to show:

If X is $\mathbb{R}^n$ and

$$d: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0} \quad \text{is given by } d(x,y) = |y-x|$$

then the triangle inequality holds:

$$|x+y| \leq |x| + |y|.$$

or, if $x = p-q$ and $y = q-r$ then

$$|p-r| = |p-q+q-r| < |p-q| + |q-r|$$

so that

$$d(p,r) \leq d(p,q) + d(q,r)$$

and (d) holds.

$\mathbb{R}^n$ will be our favorite example of a metric space.

The triangle and Schwartz inequalities

The inner product on $\mathbb{R}^n$ is the function

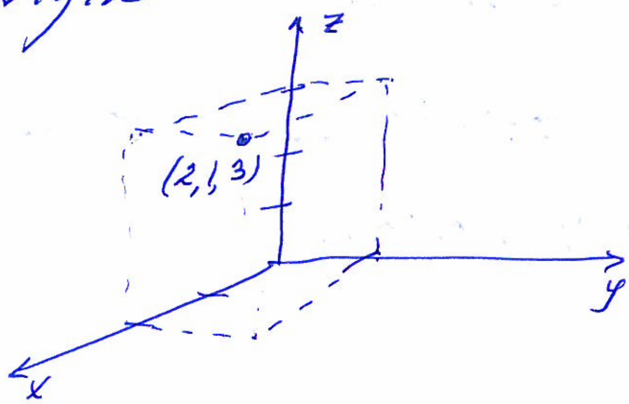
$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \langle x, y \rangle \end{aligned} \quad \text{given by}$$

$$\langle x, y \rangle = (x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

The absolute value on $\mathbb{R}^n$ is the function

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto |x| \end{aligned} \quad \text{given by } |x| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\langle x, x \rangle}.$$

Pictorially, $|x|$ is the distance from $x = (x_1, \ldots, x_n)$ to the origin



$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ and

$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$.

$\mathbb{R}^1 = \{x \mid x \in \mathbb{R}\} = \mathbb{R}$

$\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ can be identified with $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.

<u>Lagrange's identity</u> If $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$ then

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)-\left(\sum_{i=1}^n x_iy_i\right)^2=\frac{1}{2}\sum_{i,j}(x_iy_j-x_jy_i)^2.$$

<u>Proof</u>

$$\frac{1}{2}\sum_{i,j=1}^n(x_iy_j-x_jy_i)^2=\frac{1}{2}\sum_{i,j=1}^n x_i^2y_j^2-2x_iy_jx_jy_i+x_j^2y_i^2$$

$$=\frac{1}{2}\sum_{i,j=1}^n x_i^2y_j^2+\frac{1}{2}\sum_{i,j=1}^n x_j^2y_i^2-\sum_{j,i=1}^n x_iy_ix_jy_j$$

$$=\sum_{i,j=1}^n x_i^2y_j^2-\left(\sum_{i=1}^n x_iy_i\right)^2.$$

$$=\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{j=1}^n y_j^2\right)-\left(\sum_{i=1}^n x_iy_i\right)^2.$$

//

If $n=2$

$$\frac{1}{2}\left((x_1y_1-x_1y_1)^2+(x_1y_2-x_2y_1)^2+(x_2y_1-x_1y_2)^2+(x_2y_2-x_2y_2)^2\right).$$

$= \dots$

$$=(x_1^2+x_2^2)(y_1^2+y_2^2)-(x_1y_1+x_2y_2)^2.$$

<u>Theorem</u> (The Schwartz inequality) If $x, y \in \mathbb{R}^n$ then

$$\langle x, y\rangle \le |x||y|.$$

<u>Proof</u> Lagrange's identity tells us

$$|x|^2|y|^2 - \langle x, y\rangle^2 \ge 0.$$

So $(|x||y|)^2 \ge \langle x, y\rangle^2$.

So $|x||y| \ge \langle x, y\rangle$. //.

<u>Theorem</u> (The triangle inequality) Let $x, y \in \mathbb{R}^n$.

Then

$$|x+y| \le |x| + |y|.$$

<u>Proof</u>

$$\begin{aligned}\langle x+y, x+y\rangle &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= |x|^2 + 2\langle x, y\rangle + |y|^2 \\ &\le |x|^2 + 2|x||y| + |y|^2 \\ &= (|x| + |y|)^2\end{aligned}$$

So $|x+y|^2 \le (|x|+|y|)^2$

So $|x+y| \le |x| + |y|$. //.

Note that Lagrange's identity works with $\mathbb{R}$ replaced by any field,
and the Schwartz and triangle inequalites are valid with $\mathbb{R}$ replaced by any ordered field.