A **field** is a set $\mathbb{F}$ with operations

$$+: \mathbb{F}\times\mathbb{F} \to \mathbb{F} \qquad \text{and} \qquad \mathbb{F}\times\mathbb{F} \to \mathbb{F}$$
$$(a,b) \mapsto a+b \qquad\qquad (a,b) \mapsto ab$$

such that

(a) If $a,b,c \in \mathbb{F}$ then $(a+b)+c = a+(b+c)$,

(b) If $a,b \in \mathbb{F}$ then $a+b = b+a$,

(c) There exists $0 \in \mathbb{F}$ such that

if $a \in \mathbb{F}$ then $0+a = a$ and $a+0 = a$,

(d) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that

$a+(-a) = 0$ and $(-a)+a = 0$,

(e) If $a,b,c \in \mathbb{F}$ then $(ab)c = a(bc)$,

(f) If $a,b,c \in \mathbb{F}$ then

$(a+b)c = ac+bc$ and $c(a+b) = ca+cb$,

(g) There exists $1 \in \mathbb{F}$ such that

if $a \in \mathbb{F}$ then $1\cdot a = a$ and $a\cdot 1 = a$

(h) If $a \in \mathbb{F}$ and $a \neq 0$ then there exists $a^{-1} \in \mathbb{F}$

such that $aa^{-1} = 1$ and $a^{-1}a = 1$,

(i) If $a,b \in \mathbb{F}$ then $ab = ba$.

Example 1 Let $\mathbb{F}$ be a field and let $a \in \mathbb{F}$.
Show that $a \cdot 0 = 0$.

Proof
$$a \cdot 0 = a \cdot (0+0), \quad \text{by (c)}$$
$$= a \cdot 0 + a \cdot 0, \quad \text{by (f)}.$$
Add $-a \cdot 0$ to each side and use (d) to get
$$0 = a \cdot 0. //$$

Example 2 Let $\mathbb{F}$ be a field and let $a \in \mathbb{F}$.
Show that $-(-a) = a$

Proof By (d),
$$-(-a) + (-a) = 0 = a + (-a).$$
Add a to each side and use (d) to get
$$-(-a) = a. //$$

Example 3 Let $\mathbb{F}$ be a field and let $a \in \mathbb{F}$ with $a \neq 0$.
Show that $(a^{-1})^{-1} = a$.

Proof By (h),
$$(a^{-1})^{-1} \cdot a^{-1} = 1 = a \cdot a^{-1}.$$
Multiply each side by a and use (h) and (g) to get
$$(a^{-1})^{-1} = a. //.$$

Example 4 Let $\mathbb{F}$ be a field and let $a \in \mathbb{F}$.

Show that $a(-1) = -a$.

Proof By (f),

$$a(-1) + a \cdot 1 = a(-1+1) = a \cdot 0 = 0,$$

where the last equality follows from Example 1.

So, by (g), $a(-1) + a = 0$.

Add $-a$ to each side and use (d) and (c) to get

$$a(-1) = -a. \quad //$$

Example 5 Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$.

Show that $(-a)b = -ab$.

Proof

$$\begin{aligned} (-a)b + ab &= (-a+a)b, \quad \text{by (f)} \\ &= 0 \cdot b, \quad \text{by (d)} \\ &= 0, \quad \text{by Example 1.} \end{aligned}$$

Add $-ab$ to each side and use (d) and (c) to get

$$(-a)b = -ab. \quad //$$

Example 6 Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$.

Show that $(-a)(-b) = ab$.

Proof

$$\begin{aligned} (-a)(-b) &= -(a(-b)), \quad \text{by Example 5,} \\ &= -(-ab), \quad \text{by Example 5,} \\ &= ab, \quad \text{by Example 2.} \quad // \end{aligned}$$

Let S be a set.

A <u>total order</u> on S is a relation $\le$ on S such that

(a) If $x,y,z \in S$ and $x \le y$ and $y \le z$ then $x \le z$,

(b) If $x,y \in S$ and $x \le y$ and $y \le x$ then $x = y$,

(c) If $x,y \in S$ then $x \le y$ ~~and~~ or $y \le x$.

An <u>ordered field</u> is a <u>field</u> $\mathbb{F}$ with a total order $\le$ such that

(a) If $a,b,c \in \mathbb{F}$ and $a \le b$ then $a+c \le b+c$,

(b) If $a,b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

<u>Example 1</u> Let $\mathbb{F}$ be an ordered field and let $a \in \mathbb{F}$ with $a > 0$. Show that $-a < 0$.

<u>Proof.</u> Assume $a \in \mathbb{F}$ and $a > 0$.

Then $a + (-a) > 0 + (-a)$, by (OFb).

So $0 > -a$, by (Fd) and (Fc). //.

<u>Example 2</u> Let $\mathbb{F}$ be an ordered field and let $a \in \mathbb{F}$ with $a \ne 0$. Show that $a^2 > 0$.

<u>Proof</u> <u>Case 1</u>: $a > 0$.

Then $a \cdot a > a \cdot 0$, by (OFb).

So $a^2 > 0$, by F Example 1.

Case 2 $a<0$.

Then $-a>0$, by Example 1.

Then $(-a)^2>0$, by Case 1.

So $a^2>0$, by FExample 6 //.

Example 3 Let $\mathbb{F}$ be an ordered field. Show that $1\geq 0$.

Proof By (Fq),

$$1=1^2\geq 0, \text{ by Example 2 //.}$$

Example 4 Let $\mathbb{F}$ be an ordered field. Let $a\in\mathbb{F}$ with $a>0$. Show that $a^{-1}>0$.

Proof Assume $a\in\mathbb{F}$ and $a>0$.

By Example 2, $a^{-2}=(a^{-1})^2>0$.

So $a(a^{-1})^2>a\cdot 0$, by (OFb).

So $a^{-1}>0$, by (Fh) and FExample 1 //.

Example 5 Let $\mathbb{F}$ be an ordered field and let $a,b\in\mathbb{F}$. Show that if $a\geq 0$ and $b\geq 0$ then $a+b\geq 0$.

$$\begin{aligned} a+b &\geq 0+b, \text{ by (OFa)} \\ &\geq 0+0, \text{ by (OFa)} \\ &=0, \text{ by (Fc) //.} \end{aligned}$$

Example 6 Let $\mathbb{F}$ be an ordered field and let $a, b \in \mathbb{F}$.
Show that if $0 < x < y$ then $y^{-1} < x^{-1}$.

Proof Assume $0 < x < y$.

So $x > 0$ and $y > 0$.

Then, by Example 4, $x^{-1} > 0$ and $y^{-1} > 0$.

Thus, by (OFb) $x^{-1}y^{-1} > 0$.

Since $x < y$, then $y - x > 0$, by (OFa)

So, by (OFb), $x^{-1}y^{-1}(y-x) > 0$.

So, by (Fh), $x^{-1} - y^{-1} > 0$.

So, by (OFa), $x^{-1} > y^{-1}$. //

Proposition Let $\mathbb{F}$ be an ordered field. Let $x, y \in \mathbb{F}$ with $x \geq 0$ and $y \geq 0$. Then

$$x \leq y \text{ if and only if } x^2 \leq y^2.$$

Proof Assume $x, y \in \mathbb{F}$ and $x \geq 0$ and $y \geq 0$.

To show: (a) If $x \leq y$ then $x^2 \leq y^2$

(b) If $x^2 \leq y^2$ then $x \leq y$.

(a) Assume $y \geq x$.

Then $y - x \geq 0$ and, since $x \geq 0$ and $y \geq 0$ then $x + y \geq$.

So $(y-x)(x+y) \geq 0 \cdot (x+y)$

So $y^2 - x^2 \geq 0$

So $y^2 \geq x^2$.

(b) Assume $x^2 \leq y^2$.

Then $y^2 + (-x^2) \geq x^2 + (-x^2) = 0$.

So $y^2 - x^2 \geq 0$.

So $(y-x)(y+x) \geq 0$.

Since $x \geq 0$ and $y \geq 0$ then $x+y \geq 0$.

Case 1: ~~$x \geq 0$ and $y \geq 0$~~. $x+y \neq 0$

Then $x+y > 0$ and $(x+y)^{-1} > 0$.

So $(y-x)(y+x)(y+x)^{-1} \geq 0(x+y)^{-1} = 0$.

So $y - x \geq 0$.

So $y \geq x$

Case 2 ~~$x = 0$ and $y \geq 0$~~. $x+y=0$.

Then $x=0$ and $y=0$ (since $x \geq 0$ and $y \geq 0$).

So $y \geq x$. //.