620-295 Real Analysis with applications lect 25, 3 May 2010. (1)

A **field** is a set $\mathbb{F}$ with operations

$$+: \mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F} \qquad \text{and} \qquad \mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F}$$
$$(a,b) \longmapsto a+b \qquad\qquad (a,b) \longmapsto ab$$

such that

(a) If $a,b,c\in\mathbb{F}$ then $(a+b)+c = a+(b+c)$,

(b) If $a,b\in\mathbb{F}$ then $a+b=b+a$,

(c) There exists $0\in\mathbb{F}$ such that

if $a\in\mathbb{F}$ then $0+a=a$ and $a+0=a$,

(d) If $a\in\mathbb{F}$ then there exists $-a\in\mathbb{F}$ such that

$a+(-a)=0$ and $(-a)+a=0$,

(e) If $a,b,c\in\mathbb{F}$ then $(ab)c=a(bc)$,

(f) If $a,b,c\in\mathbb{F}$ then

$(a+b)c = ac+bc$ and $c(a+b)=ca+cb$,

(g) There exists $1\in\mathbb{F}$ such that

if $a\in\mathbb{F}$ then $1\cdot a=a$ and $a\cdot 1=a$

(h) If $a\in\mathbb{F}$ and $a\neq 0$ then there exists $a^{-1}\in\mathbb{F}$

such that $aa^{-1}=1$ and $a^{-1}a=1$,

(i) If $a,b\in\mathbb{F}$ then $ab=ba$.

The positive integers is the set

$$\mathbb{Z}_{>0} = \{1, 1+1, 1+1+1, 1+1+1+1, \ldots\}$$

with addition given by concatenation so that

$$(1+1)+(1+1+1) = 1+1+1+1+1, \text{ for example.}$$

The counting by k function is $m_k : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ given by

$$m_k(1) = k \quad \text{and} \quad m_k(a+b) = m_k(a) + m_k(b).$$

For example:

$$\begin{array}{ccc} m_5 : \mathbb{Z}_{>0} & \longrightarrow & \mathbb{Z}_{>0} \\ 1 & \longmapsto & 5 \\ 2 & \longmapsto & 10 \\ 3 & \longmapsto & 15 \\ \vdots & & \vdots \end{array}$$

Multiplication on $\mathbb{Z}_{>0}$ is given by

$$\begin{array}{ccc} \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} & \longrightarrow & \mathbb{Z}_{0} \\ (k, l) & \longmapsto & m_k(l) \end{array}$$

The nonegative integers is the set

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\} \quad \text{where}$$

2 really means $1+1$, 3 really means $1+1+1$, etc.

The integers is the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$$

with addition and multiplication determined from $\mathbb{Z}_{>0}$ and the axioms of a field, except (h).

Define a relation $\leq$ on $\mathbb{Z}$ by

$$x \leq y \text{ if there exists } n \in \mathbb{Z}_{\geq 0} \text{ such that } x+n=y.$$

The <u>rational numbers</u> is the set

$$\mathbb{Q} = \left\{ \tfrac{a}{b} \mid a,b \in \mathbb{Z},\ b \neq 0 \right\}$$

with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$,

and $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

The <u>real numbers</u> $\mathbb{R}$ is the set

$$\mathbb{R} = \mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 0}, \quad \text{where}$$

$$\mathbb{R}_{\geq 0} = \{ a_\ell a_{\ell-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \mid \ell \in \mathbb{Z}_{\geq 0},\ a_i \in \{0,1,\ldots,9\} \}$$

$$\mathbb{R}_{\leq 0} = \{ -a_\ell a_{\ell-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots \mid \ell \in \mathbb{Z}_{\geq 0},\ a_i \in \{0,1,\ldots,9\} \}$$

and

$$a = b \text{ if } a - b = 0.000\ldots$$

where <u>addition</u> and <u>multiplication</u> are given by

$a+b$ to an accuracy of k decimal places is

$a_{\geq(-k-1)} + b_{\geq(-k-1)}$ (addition in $\mathbb{Q}$),

with $a_{\geq -k} = a_\ell a_{\ell-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_{-k} 000\ldots$

and ab to an accuracy of k decimal places is

$a_{\geq(-m-k)} \cdot b_{\geq(-\ell-k)}$ (multiplication in $\mathbb{Q}$)

The complex numbers $\mathbb{C}$ is the set

$$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\} \text{ with } i^2 = -1,$$

the addition and multiplication on $\mathbb{R}$ and the properties of a field determine the addition ~~and~~ multiplication in $\mathbb{C}$.

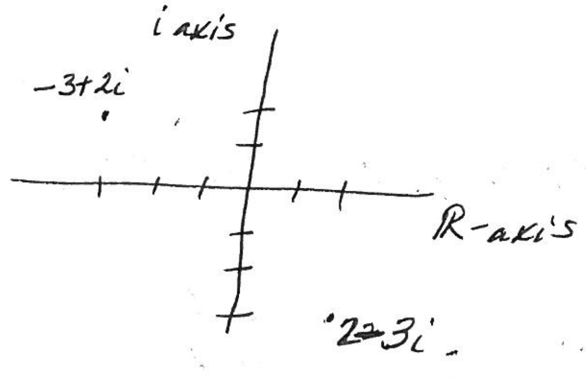
For example:

$$(3+2i) + (5+7i) = 8+9i$$

and $$(3+2i)(5+7i) = 15 + 21i + 10i + 14i^2$$
$$= (15-14) + 31i = 1 + 31i$$

and $$(3+2i)^{-1} = \frac{1}{(3+2i)}\frac{(3-2i)}{(3-2i)} = \frac{1}{9-4}(3-2i) = \frac{3}{5} - \frac{2}{5}i.$$

Graphing:



Example Let $a = 210.98765432109876543210987\ldots$

and $b = 100101.101001000100001\ldots$

Compute $a+b$ and $a \cdot b$ to 10 decimal places.

Well

$$\begin{array}{r} 210.98765432109876\,5\ldots \\ 100101.10100100010000\,1\ldots \\ \hline 100312.08865532119876\,6 \end{array}$$

or 7

and $b = 10^5 + 10^2 + 1 + 10^{-1} + 10^{-3} + 10^{-6} + 10^{-10} + 10^{-15} + 10^{-21} + \ldots$

So

$$\begin{array}{rr} 10^5 a = & 21098765.43210987654321\,0\ldots \\ 10^2 a = & 21098.76543210987654\,3\ldots \\ \hline (10^5+10^2)a = & 21119864.14754198641975\,3\ldots \\ a = & 210.98765432109876\,5\ldots \\ \hline (10^5+10^2+1)a = & 21120075.18519530751851\,8\ldots \\ 10^{-1}a = & 21.09876543210987\,6\ldots \\ \hline & 21120096.28396073962839\,4 \end{array}$$

etc.